An Algorithm for Decomposing Multivariate Hypergeometric Terms

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Outline

- Multivariate hypergeometric terms
- The Ore–Sato theorem
- An algorithm for the Ore–Sato decomposition

Multivariate hypergeometric terms

Let $\mathbb{F}$ be a field of characteristic zero and $x_1, \ldots, x_m$ indets. A function $H(x_1, \ldots, x_m)$ is hypergeometric over $\mathbb{F}(x_1, \ldots, x_m)$ if the i-th shift quotient

$$r_i = \frac{H(x_1, \ldots, x_i + 1, \ldots, x_n)}{H(x_1, \ldots, x_i, \ldots, x_m)} \in \mathbb{F}(x_1, \ldots, x_m), \quad i = 1, \ldots m.$$

Satisfy the compatibility conditions:

$$\frac{r_i(x_1, \ldots, x_j + 1, \ldots, x_m)}{r_i(x_1, \ldots, x_j, \ldots, x_m)} = \frac{r_j(x_1, \ldots, x_i + 1, \ldots, x_m)}{r_j(x_1, \ldots, x_i, \ldots, x_m)}$$

for $i, j \in \{1, \ldots, m\}$

Example. Let n and k be indets.

$$H(n,k) = \frac{1}{n^2+k^2}\binom{n}{k}^2.$$

$$\frac{H(n+1,k)}{H(n,k)} = \frac{(n^2+k^2)(n+1)^2}{(n-k+1)^2(n^2+2n+1+k^2)}$$

and

$$\frac{H(n,k+1)}{H(n,k)} = \frac{(n-k)^2(n^2+k^2)}{(k+1)^2(n^2+k^2+2k+1)}.$$

Factorial terms

- A polynomial $f \in \mathbb{F}[x_1, \ldots, x_m]$ is said to be integer-linear if

  $\exists\ v_1, \ldots, v_m \in \mathbb{Z},$ and $P \in \mathbb{F}[y],\ f = P\left(v_1 x_1 + \cdots + v_m x_m\right).$

- A hypergeometric term $T(x_1, \ldots, x_m)$ is called a factorial term if

$$\forall i \in \{1, \ldots, m\},\ \frac{T(x_1, \ldots, x_i + 1, \ldots, x_n)}{T(x_1, \ldots, x_i, \ldots, x_m)}$$

  is the product of the powers of integer-linear polynomials.

Example. Let n and k be indets.

$$T(n,k) = \binom{n}{k}^2.$$

$$\frac{T(n+1,k)}{T(n,k)} = \frac{(n+1)^2}{(n-k+1)^2}$$

and

$$\frac{T(n,k+1)}{T(n,k)} = \frac{(n-k)^2}{(k+1)^2}.$$

Thus, T is a factorial term with shift quotients as above.

Testing integer-linear

Proposition. Let $f \in \mathbb{F}[x_1, \ldots, x_m]$. Write

$$f = h_0 + \ldots + h_k$$

where h_i is homogeneous of degree d_i, $i = 0, \ldots, k$. Then f is integer-linear iff there are integers $v_1, \ldots, v_m$ s.t.

$$h_i = c_i \left(v_1 x_1 + \ldots + v_m x_m\right)^{d_i} \quad \text{for some } c_i \in \mathbb{F},$$

where $i = 0, \ldots, k$.

Ore-Sato theorem

Theorem. For a hypergeometric term H, $\exists\ f \in \mathbb{F}(x_1, \ldots, x_m)$ and a factorial term T s.t.

$$H = f\,T. \qquad (*)$$

Moreover, $(*)$ is unique up to a constant if none of the nontrivial irreducible factors of f is integer-linear.

Remark. H is said to be proper if f in $(*)$ is a polynomial.

Structure of factorial terms

Payne's Product. Let a and b be two integers.

$$\prod_{i}{}_{a}^{b} = \begin{cases} \prod_{i=a}^{b-1} & \text{if } a < b \\ 1 & \text{if } a = b \\ 1/\prod_{i=b}^{a-1} & \text{if } a > b. \end{cases}$$

Lemma. Let $T(x_1, \ldots, x_m)$ be a factorial term. Then there exist

- a finite subset $V \subset \mathbb{Z}^m \setminus \{0\}$,
- $r_{\mathbf{v}} \in \mathbb{F}(y) \setminus \mathbb{F}$ for each $\mathbf{v} \subset V$
- $c_i \in \mathbb{F} \setminus \{0\}$, $i = 1, \ldots m$,

s.t.

$$\frac{T(x_1, \ldots x_i + 1, \ldots x_m)}{T(x_1, \ldots, x_i, \ldots, x_m)} = c_i \prod_{\mathbf{v} \in V} \prod_{j}{}_{0}^{v_i} r_{\mathbf{v}}(v_1 x_1 + \cdots + v_m x_m + j)$$

with $\mathbf{v} = (v_1, \ldots, v_m)$.

Computing the Ore-Sato decomposition

Given the shift quotients $r_1, \dots r_m$ of a hypergeometric term $H(x_1, \dots, x_m)$,

compute $f \in \mathbb{F}(x_1, \dots, x_m)$ and the shift quotients $t_1, \dots t_m$ of a factorial term T s.t.

$$H = fT$$

is the canonical Ore-Sato decomposition of H.

Outline of the algorithm

1. (Factorization) For $i = 1, \ldots, m$, compute $f_i, t_i \in \mathbb{F}(x_1, \ldots x_m)$ s.t. $r_i = f_i t_i$, where
   - f_i is monic w.r.t. a fixed monomial order and has no integer-linear factor;
   - all the irreducible factors of t_i are integer-linear.
2. (Rational part) Find a nonzero rational solution f of the system

$$z(x_1, \ldots, x_i+1, \ldots, x_m) = f_i z(x_1, \ldots, x_i, \ldots, x_m), \quad i = 1, \ldots, m.$$

3. (Integer-linear). Use the factorization of the t_i to find $V \in \mathbb{Z}^m$ s.t.

$$t_i = c_i \prod_{\mathbf{v} \in V} P_{i\mathbf{v}}(v_1 x_1 + \cdots v_m x_m) \quad \text{with } \mathbf{v} = (v_1, \ldots, v_m).$$

4. (Uniform rational). By Payne's Lemma, use $P_{i\mathbf{v}}$'s to compute $r_{\mathbf{v}}$ s.t.

$$t_i = c_i \prod_{\mathbf{v} \in V} \prod_{j}{}_0^{v_i} r_{\mathbf{v}}(v_1 x_1 + \cdots + v_m x_m + j), \quad i = 1, \ldots, m.$$

5. Return $[f,\ [c_1, \ldots, c_m],\ [(\mathbf{v}, r_{\mathbf{v}}) | \mathbf{v} \in V]]$.

Example

Consider a hypergeometric term $H(n, k)$ with

1. $$\frac{H(n+1,k)}{H(n,k)} = \frac{(n^2+k^2)}{(n^2+2n+1+k^2)} \cdot \frac{(n+1)^2}{(n-k+1)^2} = f_1 \cdot t_1$$

   and

   $$\frac{H(n,k+1)}{H(n,k)} = \frac{(n^2+k^2)}{(n^2+k^2+2k+1)} \cdot \frac{(n-k)^2}{(k+1)^2} = f_2 \cdot t_2$$

2. Compute $f = \frac{1}{n^2+k^2}$ s.t. $\frac{f(n+1,k)}{f(n,k)} = f_1$ and $\frac{f(n,k+1)}{f(n,k)} = f_2$

$$H = \frac{1}{n^2+k^2} T,$$ where T is a factorial term.

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2. Compute $f = \frac{1}{n^2+k^2}$ s.t. $\frac{f(n+1,k)}{f(n,k)} = f_1$ and $\frac{f(n,k+1)}{f(n,k)} = f_2$

$$H = \frac{1}{n^2+k^2} T,$$ where T is a factorial term. $\Longrightarrow$ H is not proper.

3. Compute $\mathbf{v}_1 = (1, -1), \mathbf{v}_2 = (1, 0), \mathbf{v}_3 = (0, 1)$ s.t.
$t_1 = \frac{(n+1)^2}{(n-k+1)^2}$ and $t_2 = \frac{(n-k)^2}{(k+1)^2}$

4. Compute $r_{\mathbf{v}_1} = \frac{1}{(1+X)^2}, r_{\mathbf{v}_2} = (1+X)^2, r_{\mathbf{v}_3} = \frac{1}{(1+X)^2}$ s.t.
$t_1 = 1 \cdot r_{\mathbf{v}_1}(n-k) r_{\mathbf{v}_2}(n)$ and $t_2 = 1 \cdot \frac{r_{\mathbf{v}_3}(k)}{r_{\mathbf{v}_1}(n-k-1)}$

5. Return $[f, [1,1], [[\mathbf{v}_1, r_{\mathbf{v}_1}], [\mathbf{v}_2, r_{\mathbf{v}_2}], [\mathbf{v}_3, r_{\mathbf{v}_3}]]]$

$$T = \left(\frac{(1)_{n+1}}{(1)_{n-k+1}(1)_{k+1}} \right)^2 = \binom{n}{k}^2$$

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$t_1 = 1 \cdot r_{\mathbf{v}_1}(n-k) r_{\mathbf{v}_2}(n)$ and $t_2 = 1 \cdot \frac{r_{\mathbf{v}_3}(k)}{r_{\mathbf{v}_1}(n-k-1)}$

5. Return $[f, [1, 1], [[\mathbf{v}_1, r_{\mathbf{v}_1}], [\mathbf{v}_2, r_{\mathbf{v}_2}], [\mathbf{v}_3, r_{\mathbf{v}_3}]]]$

$$T = \left(\frac{(1)_{n+1}}{(1)_{n-k+1}(1)_{k+1}}\right)^2 = \binom{n}{k}^2 \implies H = \frac{1}{n^2+k^2} \cdot \binom{n}{k}^2.$$

Summary

Results.

1. An algorithm for the Ore-Sato decomposition
2. A criterion concerning proper hypergeometric terms

Future work.

- Compute multiplicative decomposition of hypergeometric terms with parameters.
- Devise an algorithm for decomposing multivariate hyperexponential functions.

Thank you!