Convergent Expansions and Bounds for the Incomplete Elliptic Integral of the Second Kind near the Logarithmic Singularity

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Background

Let $a, b \in \mathbb{R}$ with $a \geq b$. Consider the ellipse $\mathcal{C}$:

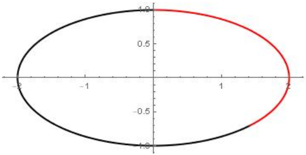
$$x = a \sin\theta, \quad y = b \sin\theta, \quad 0 \leq \theta < 2\pi.$$

The arc length of $\mathcal{C}$ for $0 \leq \theta \leq \phi$ is

$$L = \int_0^{\phi} \sqrt{x'(\theta)^2 + y'(\theta)^2} d\theta$$

$$= a \int_0^{\phi} \sqrt{1 - k^2 \sin^2\theta d} d\theta, \quad (1)$$

where $k = \sqrt{1 - (b/a)^2}$
is the eccentricity of $\mathcal{C}$.



arc length of an ellipse

Background

Set $\lambda = \sin\phi$ and $t = \sin\theta$ in (1). Then

$$L = aE(\lambda, k),$$

where

$$E(\lambda, k) = \int_0^\lambda \frac{\sqrt{1 - k^2 t^2}}{\sqrt{1 - t^2}} dt$$

is the Lengendre's second incomplete elliptic integral (EI) with $(\lambda, k) \in (0, 1)^2$.

- The domain of $E(\lambda, k)$ can be extended to the bi-disk $|\lambda| < 1, |k| < 1$ of $\mathbb{C}^2$ by analytic continuation.
- $E(\lambda, k)$ is not elementary integrable by Liouville's theorem.

Background

- The reflection-type relation:

$$E(\lambda, k) = E(k) - \sqrt{1-k^2} \cdot E\left(\sqrt{1-\lambda^2}, \sqrt{-k^2/(1-k^2)}\right), \tag{2}$$

where $E(k) = E(1, k)$ is the second complete EI.

Motivation

- Vel (1969), Kaplan (1948): Asymptotic expansions for $E(\lambda, k)$ near the logarithmic singularity.
- Carlson-Gustafson (1985, 1994), López (2000, 2001): Asymptotic expansions for symmetric elliptic integrals, which are equivalent to the Lengendre's forms, with various parametric regimes.
- Karp-Sitnik (2007): Asymptotic approximations for the first and third incomplete EI near logarithmic singularity.

Motivation

Our contribution:

- Two series expansions for $E(\lambda, k)$ which are asymptotic when λ or k or both tend to 1 and converges at each point of $(0,1)^2$ by using symbolic summation and hypergeometric inequalities.

- High-precision approximations for $E(\lambda, k)$ which are much better than those in the literature.

Expansion of Byrd-Friedman Revisited

Let $(\lambda, k) \in (0,1)^2$. Set $\beta = (1-\lambda^2)/(1-k^2)$.
Theorem 1 (Bryd-Friedman, 1971). Suppose $\beta > \lambda^2$. For $N \geq 1$, we have

$$E(\lambda, k) = \lambda \sum_{j=0}^{N} (-1)^j \frac{(-1/2)_j}{j!} \left[\frac{\lambda^2}{\beta}\right]^j$$
$$+ \ln\left(\frac{1-\lambda}{1+\lambda}\right) \sum_{j=1}^{N} \frac{(-1/2)_j (1/2)_j}{j!(j-1)!} (1-k^2)^j$$
$$+ \frac{1}{\lambda} \sum_{j=1}^{N} \left[\frac{\lambda^2}{\beta}\right]^j \sum_{n=0}^{j-1} (-1)^{j+n-1} \frac{(-1/2)_j (1/2-j)_n}{j!(1-j)_n} \left(\frac{1-\lambda^2}{\lambda^2}\right)^n + R_N(\lambda, k). \tag{3}$$

The remainder $R_N(\lambda, k)$ satisfies

$$|R_N(\lambda, k)| \leq \frac{\lambda(1-\lambda^2)(2N-1)!!}{N 2^{N+2} (N+1)!} \left[\frac{\lambda^2}{\beta}\right]^{N+1}.$$

Expansion of Byrd-Friedman Revisited

Ingredients for the proof: Set $k'^2 = 1 - k^2$. Then

$$E(\lambda, k) = \int_0^\lambda \sqrt{\frac{1-k^2t^2}{1-t^2}}dt = \int_0^\lambda dt \left(1 + \frac{k'^2t^2}{1-t^2}\right)^{1/2}.$$

Expand $\left[1 + (k'^2t^2)/(1-t^2)\right]^{1/2}$ into the binomial series, integrate term-wise for the integrand, and apply alternating series test to deduce an upper bound for $|R_N(\lambda, k)|$.

- The expansion (3) does not converge for the whole unit square $(0,1)^2$.
- The rate of convergence for (3) is not satisfactory.

The First Asymptotic Expansion

For $n \geq 0$, set

$$s_n(x) = \sum_{j=n+1}^{\infty} \frac{(-1/2)_j (1/2 - j)_n}{j!(1-j)_n} (-x)^j.$$

Using the method of creative telescoping, we get

Lemma 1. For $n \geq 0$, $s_n(x)$ is an elementary function in x and satisfies

$$\begin{aligned} 4(n+2)(n+3)s_{n+3}(x) = a_n(x)s_{n+2}(x) + b_n(x)s_{n+1}(x) \\ + c_n(x)s_n(x) + d_n(x), \end{aligned}$$

where $a_n(x), b_n(x), c_n(x), d_n(x)$ are polynomials in x with hypergeometric coeffs in n.

The First Asymptotic Expansion

Theorem 2 (Karp-Z., 2023) . For $(\lambda, k) \in (0,1)^2$ and $N \geq 1$, we have

$$E(\lambda, k) = \lambda\sqrt{1+\frac{\lambda^2}{\beta}} + \ln\left(\frac{1-\lambda}{1+\lambda}\right)\sum_{j=1}^{N}\frac{(-1/2)_j(1/2)_j}{j!(j-1)!}(1-k^2)^j$$
$$-\frac{1}{\lambda}\sum_{n=0}^{N-1}\left(\frac{1-\lambda^2}{-\lambda^2}\right)^n s_n\left(\frac{\lambda^2}{\beta}\right) + \tilde{R}_N(\lambda, k), \tag{4}$$

where $\beta = (1-\lambda^2)/(1-k^2)$ and $s_n(x)$ is given in Lemma 1. Moreover, $\tilde{R}_N(\lambda, k)$ is negative and satisfies

The First Asymptotic Expansion

$$\frac{(1/2)_N(1/2)_{N+1}(1-k^2)^N}{2N!(N+1)!} f_{N+1}(\lambda,k) < -\tilde{R}_N(\lambda,k)$$
$$< \frac{(1/2)_N(1/2)_{N+1}(1-k^2)^N}{2N!(N+1)!} f_N(\lambda,k), \quad (5)$$

where $f_N(\lambda,k)$ is increasing in N and bounded on each subset of $(0,1)^2$ with some conditions.

Idea for the proof:

- Exchange the order of the double sum of expansion for $E(\lambda,k)$ in Theorem 1 and apply Lemma 1 to deduce (4).
- Use symbolic summation algorithms to write $\tilde{R}_N(\lambda,k)$ in terms of the integral of the hypergeometric function ${}_3F_2$.
- Apply inequalities of ${}_3F_2$ to derive bounds for $\tilde{R}_N(\lambda,k)$.

The First Asymptotic Expansion

- The error bound (5) implies that expansion (4) is convergent at any point in $(0, 1)^2$ and is asymptotic as (λ, k) approaches the logarithmic singularity $(1, 1)$.
- Based on Theorem 2, we can use the reflection-type relation (2) for $E(\lambda, k)$ to derive another series expansion.

Numerical Experiments

- The first order approximation for $E(\lambda, k)$ given by Theorem 2 is

$$E_1(\lambda, k) = \left(\lambda - \frac{1}{\lambda}\right)\sqrt{1 + \frac{\lambda^2(1-k^2)}{1-\lambda^2}} - \frac{1-k^2}{4}\ln\frac{1-\lambda}{1+\lambda} + 1/\lambda.$$

- The first order approximation for $E(\lambda, k)$ given by Carlson-Gustafson and López is

$$\bar{E}_1(\lambda, k) = \lambda(1-k^2\lambda^2)\ln\frac{4}{\sqrt{1-\lambda^2}+\sqrt{1-k^2\lambda^2}} + k^2\lambda^3.$$

Numerical Experiments

λ	k	$E(\lambda, k)$	First order approx. E_1	First order approx. $\bar{E}_1$	Relative error e_1	Relative error $\bar{e}_1$
.8	.8	.8501	.8714	.8343	-.02504	.01864
.9	.9	.9504	.9669	1.0127	-.01734	-.06551
.95	.95	.9900	1.0003	1.0704	-.01044	-.08121
.99	.99	1.0056	1.0081	1.1178	-.002531	-.04136
.95	.99	.9586	.9591	1.0472	$-.5674 \times 10^{-3}$	-.08413
.99	.999	.9916	.9916	1.0434	$-.3417 \times 10^{-4}$	-.003387

Conclusion

- Derive two series expansions for $E(\lambda, k)$ which are asymptotic when λ or k or both tend to 1 and converges at each point of $(0,1)^2$ by using symbolic summation and hypergeometric inequalities.

- Numerical experiments suggest that the corresponding high-precision approximations for $E(\lambda, k)$ which are much better than those in the literature.

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Thanks!