

On Sequences Associated to the Invariant Theory of Rank Two Lie Algebras

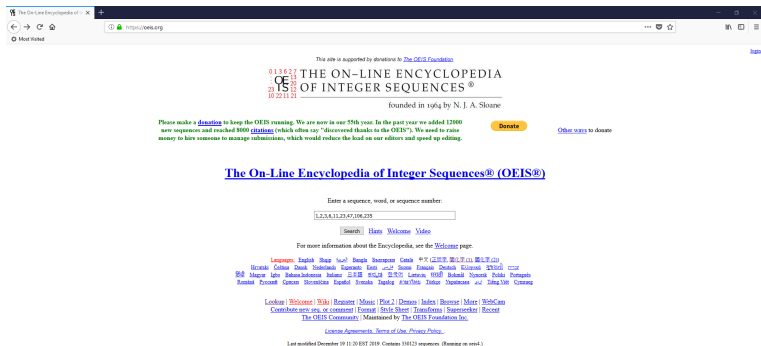
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Joint work with Alin Bostan, Jordan Tirrell and Bruce Westbury



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Last modified December 19 11:20 EST 2018. Contains 338123 sequences. (Running on oeis4.)

OEIS is an online database of integer sequences, such as Fibonacci numbers ([A000045](#)), Catalan numbers ([A000108](#)).

Two families of sequences in OEIS

a (OEIS tag)	0	1	2	3	4	5
A059710	1	0	1	1	4	10
A108307	1	1	2	5	15	51
A108304	1	2	5	15	52	202

The first family of sequences (**octant sequences**)

a (OEIS tag)	0	1	2	3	4	5
A151366	1	0	2	2	12	30
A236408	1	1	3	9	33	131
A001181	1	2	6	22	92	422
A216947	1	3	11	49	221	1113

The second family of sequences (**quadrant sequences**)

- ▶ Those sequences are associated to the invariant theory of the exceptional simple Lie algebra G_2 of rank 2.
- ▶ The quadrant sequences are related to the octant sequences by the branching rules for $SL(3)$ of G_2 .

Octant sequences

The first family of sequences can be interpreted as lattice walks restricted to the octant. We call them **octant sequences**.

- ▶ **A059710**: enumerates the multiplicities of the trivial representation in the tensor powers of V , which is the 7-D fundamental representation of G_2 .
- ▶ **A108307**: enumerates **enhanced** 3-noncrossing set partitions.
- ▶ **A108304**: enumerates 3-noncrossing set partitions.

(Lin, 2018; Gil and Tirrell, 2019): **A108307** and **A108304** are related by the binomial transform.

Motivation and Contribution

(Bostan, Tirrell, Westbury and Z., 2019): [A059710](#) and [A108307](#) are also related by the binomial transform.

Mihailovs' conjecture: Let $T_3(n)$ be the n -th term of [A059710](#). Then T_3 is determined by $T_3(0) = 1$, $T_3(1) = 0$, $T_3(2) = 1$ and

$$14(n+1)(n+2)T_3(n) + (n+2)(19n+75)T_3(n+1) + 2(n+2)(2n+11)T_3(n+2) - (n+8)(n+9)T_3(n+3) = 0.$$

(Bostan, Tirrell, Westbury and Z., 2019): Three independent proofs of Mihailovs' conjecture.

- ▶ Two proofs are based on binomial relation between [A059710](#) and [A108307](#), together with a result by Bousquet-Mélou and Xin.
- ▶ The third one is a direct proof by the method of algebraic residues, which leads to closed formulae for the generating function of T_3 in terms of hypergeometric functions.

Quadrant sequences

The second family of sequences can be interpreted as lattice walks restricted to the quadrant. We call them **quadrant sequences**.

- ▶ [A151366](#): enumerates nonpositive bipartite trivalent graphs.
- ▶ [A236408](#): enumerates pasting diagrams.
- ▶ [A001181](#): enumerates Baxter permutations.
- ▶ [A216947](#): enumerates 2-coloured noncrossing set partitions.

Question: What are relations between quadrant sequences?

Motivation and Contribution

(Marberg, 2013): a combinatorial proof that [A151366](#), [A001181](#), and [A216947](#) are related by binomial transforms.

(Bostan, Tirrell, Westbury and Z., 2019): Derive a uniform recurrence equation for quadrant sequences and show that they are related by binomial transform by the representation theory of simple Lie algebras.

Outline

- ▶ binomial relation between the first and second octant sequences
- ▶ Three independent proofs of Mihailovs' conjecture
- ▶ Recurrence relations for the quadrant sequences

Preliminaries

Definition 1 Let G be a reductive complex algebraic group and let V be a representation of G . The sequence associated to (G, V) , denoted \mathbf{a}_V , is the sequence whose n -th term is the multiplicity of the trivial representation in the tensor power $\otimes^n V$.

Example 1 Let V be the 7-D fundamental representation of G_2 . Then [A059710](#) is the sequence associated with (G_2, V) .

Let \mathbf{a} be a sequence with n -th term $a(n)$, the **binomial transform** of \mathbf{a} is the sequence, denoted $\mathcal{B}\mathbf{a}$, whose n -th term is

$$\sum_{i=0}^n \binom{n}{i} a(i).$$

Preliminaries

Lemma 1 Assume \mathbf{a}_V is the sequence associated to (G, V) as specified in **Definition 1**. Then $\mathbf{a}_{V \oplus \mathbb{C}} = \mathcal{B}\mathbf{a}_V$.

Lemma 2 Assume \mathbf{a} enumerates walks in a lattice, confined to a domain D , using a set of steps S . Then $\mathcal{B}\mathbf{a}$ also enumerates walks in a lattice restricted to D with steps $S \amalg \{0\}$.

Lemma 3 Let $G(t)$ be the generating function of \mathbf{a} . For $k \in \mathbb{Z}$, denote the generating function of $\mathcal{B}^k \mathbf{a}$ by $\mathcal{B}^k G$. Then

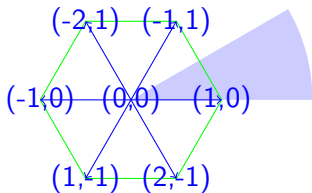
$$(\mathcal{B}^k G)(t) = \frac{1}{1 - k t} G\left(\frac{t}{1 - k t}\right).$$

Binomial relation between A059710 and A108307

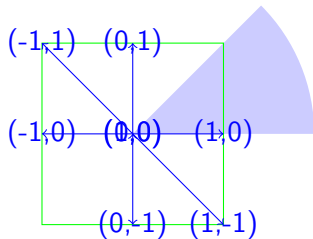
Let V be the 7-D fundamental representation of G_2 . Then

- ▶ A059710 is the sequence associated to (G_2, V) . Let $T_3(n)$ be its n -th term.
- ▶ A108307 enumerates enhanced 3-noncrossing set partitions. Let $E_3(n)$ be its n -th term.

In terms of lattice walks, we can interpret T_3 and E_3 as follows:

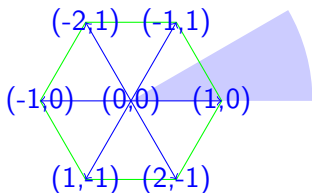


Steps in weight
lattice of G_2

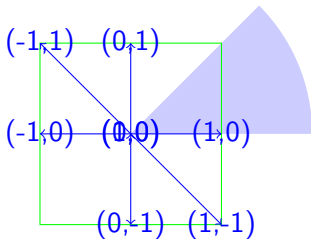


Steps in octant
related to $E_3(n)$

In terms of lattice walks, we can interpret T_3 and E_3 as follows:



Steps in weight
lattice of G_2



Steps in octant
related to $E_3(n)$

If we make a linear transformation $(x, y) \rightarrow (x + y, y)$, then it identifies the six non-zero steps, as well as the two domains.

Binomial relation between A059710 and A108307

Recall: **Lemma 2** Assume \mathbf{a} enumerates walks in a lattice, confined to a domain D , using a set of steps S . Then $\mathcal{B}\mathbf{a}$ also enumerates walks in a lattice restricted to D with steps $S \amalg \{0\}$.

Binomial relation between A059710 and A108307

Recall: **Lemma 2** Assume \mathbf{a} enumerates walks in a lattice, confined to a domain D , using a set of steps S . Then $\mathcal{B}\mathbf{a}$ also enumerates walks in a lattice restricted to D with steps $S \amalg \{0\}$.

By **Lemma 2** and the previous figures, we conclude that E_3 is the binomial transform of T_3 .

Binomial relation between A059710 and A108307

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By Lemma 2 and the previous figures, we conclude that E_3 is the binomial transform of T_3 .

(Lin, 2018; Gil and Tirrell, 2019): A108307 and A108304 are related by the binomial transform.

Recall: Lemma 1 Assume \mathbf{a}_V is the sequence associated to (G, V) as specified in Definition 1. Then $\mathbf{a}_{V \oplus \mathbb{C}} = \mathcal{B}\mathbf{a}_V$.

Thus, the octant sequences are sequences associated to

$$(G_2, V), \quad (G_2, V \oplus \mathbb{C}), \quad (G_2, V \oplus 2\mathbb{C}).$$

First proof of Mihailovs' conjecture

Mihailovs' conjecture: Let $T_3(n)$ be the n -th term of [A059710](#). Then T_3 is determined by $T_3(0) = 1$, $T_3(1) = 0$, $T_3(2) = 1$ and

$$14(n+1)(n+2)T_3(n) + (n+2)(19n+75)T_3(n+1) + 2(n+2)(2n+11)T_3(n+2) - (n+8)(n+9)T_3(n+3) = 0.$$

([Bousquet-Mélou and Xin, 2005](#)): Let $E_3(n)$ be the n -th term of [A108307](#). Then E_3 is given by $E_3(0) = E_3(1) = 1$, and

$$8(n+3)(n+1)E_3(n) + (7n^2 + 53n + 88)E_3(n+1) - (n+8)(n+7)E_3(n+2) = 0.$$

First proof of Mihailovs' conjecture

Recall: We prove that E_3 is the binomial transform of T_3 . Thus,

$$T_3(n) = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} E_3(k).$$

Set $f(n, k) = (-1)^{n-k} \binom{n}{k} E_3(k)$.

- ▶ By Bousquet-Mélou and Xin's result, $f(n, k)$ is holonomic function, which satisfies ordinary difference equations for n and k , respectively.
- ▶ **Idea:** Using creative telescoping method (Zeilberger, 1990), which is an algorithmic approach to compute a differential/difference equation for the integration/sum of holonomic functions, to drive a recurrence equation for T_3 .

First proof of Mihailovs' conjecture

- ▶ Using the Koutschan's Mathematica package `HolonomicFunctions.m` that implements Chyzak's algorithm for creative telescoping, we find exactly the recurrence equation in Mihailovs' conjecture.

Second proof of Mihailovs' conjecture

Recall: We prove that E_3 is the binomial transform of T_3 . Let $\mathcal{T}(t) = \sum_{n \geq 0} T_3(n)t^n$ and $\mathcal{E}(t) = \sum_{n \geq 0} E_3(n)t^n$. Then

$$\mathcal{T}(t) = \frac{1}{1+t} \cdot \mathcal{E}\left(\frac{t}{1+t}\right).$$

- ▶ By Bousquet-Mélou and Xin's result, we can derive an ODE for $\mathcal{E}(t)$.
- ▶ Using the closure properties of holonomic function (the sum, product and algebraic substitution of holonomic functions is still holonomic), we can derive an ODE for $\mathcal{T}(t)$ and convert it into a linear recurrence for $T_3(n)$, which is exactly the recurrence equation in Mihailovs' conjecture.

Third proof of Mihailovs' conjecture

Idea: In terms of lattice walks, we can interpret $T_3(n)$ to be the constant term of $W K^n$, where

$$K = (1 + x + y + xy + x^{-1} + y^{-1} + (xy)^{-1}),$$

and

$$W = x^{-2}y^{-3}(x^2y^3 - xy^3 + x^{-1}y^2 - x^{-2}y + x^{-3}y^{-1} - x^{-3}y^{-2} + x^{-2}y^{-3} - x^{-1}y^{-3} + xy^{-2} - x^2y^{-1} + x^3y - x^3y^2).$$

Let $\mathcal{T}(t) = \sum_{n \geq 0} T_3(n)t^n$. Then $\mathcal{T}(t)$ is the constant coefficient $[x^0y^0]$ of $W/(1-tK)$. In other words, $\mathcal{T}(t)$ is equal to the algebraic residue of $W/(xy - txyK)$, which is proportional to the contour integral of $W/(xy - txyK)$ over a cycle.

Third proof of Mihailovs' conjecture

Using creative telescoping method, we can compute a 6-th order ODE for $\mathcal{T}(t)$. Moreover, by using factorization of differential operators, we can show that $L_3(\mathcal{T}(t)) = 0$, where $\partial = \frac{d}{dt}$ and

$$L_3 = t^2 (2t + 1) (7t - 1) (t + 1) \partial^3 + 2t(t + 1) (63t^2 + 22t - 7) \partial^2 + (252t^3 + 338t^2 + 36t - 42) \partial + 28t(3t + 4).$$

Converting it into a linear recurrence for $T_3(n)$, we get exactly the recurrence equation in Mihailovs' conjecture.

Closed formulae

By factorization of the operator L_3 and algorithms for solving 2-nd order ODEs, we derive the following closed formula for $\mathcal{T}(t)$:

$$\mathcal{T}(t) = \frac{1}{30 t^5} \left[R_1 \cdot {}_2F_1 \left(\frac{1}{3}, \frac{2}{3}; \phi \right) + R_2 \cdot {}_2F_1 \left(\frac{2}{3}, \frac{4}{3}; \phi \right) + 5 P \right],$$

where

$$R_1 = \frac{(t+1)^2 (214 t^3 + 45 t^2 + 60 t + 5)}{t-1},$$

$$R_2 = 6 \frac{t^2 (t+1)^2 (101 t^2 + 74 t + 5)}{(t-1)^2},$$

and

$$\phi = \frac{27(t+1)t^2}{(1-t)^3}, \quad P = 28 t^4 + 66 t^3 + 46 t^2 + 15 t + 1.$$

Closed formulae

By elliptic curve theory, we derive an alternative formula for $\mathcal{T}(t)$:

$$\frac{P}{6t^5} + \frac{(7t-1)(2t+1)(t+1)}{360t^5} \left((155t^2 + 182t + 59)(11t+1)H(t) + (341t^3 + 507t^2 + 231t + 1)(5t+1)H'(t) \right),$$

where

$$H(t) = \frac{1}{g_2^{1/4}} \cdot {}_2F_1 \left(\frac{1}{12}, \frac{5}{12}; \frac{1728}{J} \right),$$
$$J = \frac{(t-1)^3 (25t^3 + 21t^2 + 3t - 1)^3}{t^6 (1-7t)(2t+1)^2 (t+1)^3},$$

and

$$g_2 = (t-1)(25t^3 + 21t^2 + 3t - 1).$$

Transcendence and asymptotics

Using those closed formulae, we can show that that $\mathcal{T}(t)$ is a transcendental power series and its n -th coefficient

$$T_3(n) \sim C \cdot \frac{7^n}{n}, \quad \text{where } C = \frac{4117715 \sqrt{3}}{864 \pi} \approx 2627.6.$$

Recurrence relations for quadrant sequences

Definition 2 Let \tilde{V} be the defining representation of $SL(3)$ and denote the dual by \tilde{V}^* . For $k \geq 0$, we define \mathcal{S}_k to be the sequence associated to $(SL(3), \tilde{V} \oplus \tilde{V}^* \oplus k\mathbb{C})$.

Remark: $SL(3)$ is the maximal subgroup of G_2 . Let V be the 7-D fundamental representation of G_2 . Then \mathcal{S}_k is the the sequence associated to $(SL(3), (V \oplus k\mathbb{C}) \downarrow_{SL(3)})$.

Theorem (Bostan, Tirrell, Westbury and Z., 2019): The quadrant sequences $\mathcal{S}_0, \mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$ are identical to the sequences in the second family listed in OEIS.

Lemma 4 Let \mathcal{G}_k be the generating function of \mathcal{S}_k , where $k \geq 0$. Then \mathcal{G}_k is the constant coefficient of $[x^0 y^0]$ of $W/(1 - tK)$, where

$$K = k + x + y + x^{-1} + y^{-1} + \frac{x}{y} + \frac{y}{x}$$

and

$$W = 1 - \frac{x^2}{y} + x^3 - x^2 y^2 + y^3 - \frac{y^2}{x}.$$

Recurrence relations for quadrant sequences

By [Lemma 4](#), \mathcal{S}_3 is identical to the sequence [A216947](#).

([Marberg, 2013](#)): The n -th term $C_2(n)$ of \mathcal{S}_3 is given by $C_2(0) = 1$, $C_2(1) = 3$ and

$$(n+5)(n+6) \cdot C_2(n+2) - 2(5n^2 + 36n + 61) \cdot C_2(n+1) + 9(n+1)(n+4) \cdot C_2(n) = 0.$$

By [Lemma 1](#), \mathcal{S}_k 's are related by binomial transforms. Thus, by [Lemma 3](#), the generating function of \mathcal{S}_k is

$$\mathcal{G}_k(t) = \frac{1}{1-kt} \cdot \mathcal{G}_3\left(\frac{t}{1-kt}\right)$$

where $\mathcal{G}_3(t)$ is the generating function of \mathcal{S}_3 .

Recurrence relations for quadrant sequences

Using closure properties of holonomic functions, we derive a uniform 4-th order recurrence equation for \mathcal{S}_k with k as a parameter.

By comparing the recurrence equations between \mathcal{S}_k 's and the sequences in the second family, and then checking initial terms, we show that

Corollary: The recurrence relations stated in OEIS for the sequences in the second family are true.

Summary

- ▶ A combinatorial proof of binomial relation between the first and second octant sequences
- ▶ Three independent proofs of Mihailovs' conjecture
 - ▶ Two proofs are based on binomial relation between the first and second octant sequences
 - ▶ A direct proof by the method of algebraic residues, which leads to closed formulae for the generating function of the first octant sequence
- ▶ A unified proof for recurrence relations of the quadrant sequences

Summary

- ▶ A combinatorial proof of binomial relation between the first and second octant sequences
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Thanks!