Desingularization in the q-Weyl Algebra

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Garoufalidis' conjecture

Conjecture: Let $J_K(n) \in \mathbb{Q}(q)$ be the Jones polynomial of a "colored" knot K. Then $\big(J_K(n)\big)_{n\in\mathbb{N}}$ has the following properties:

1. $(1-q^n)J_K(n)$ satisfies a bimonic recurrence relation,
2. $J_K(n)$ does not satisfy a monic recurrence relation.

- $J_K(n)$ satisfies a nonzero linear q-difference equation, *i.e.*,

  $$p_r(q,q^n)J_K(n+r)+(\cdots)J_K(n+r-1)+\cdots+p_0(q,q^n)J_K(n)=0,$$

  where $p_i(n) \in \mathbb{Q}[q,q^n]$.

- If $J_K(n)=\sum_{k=0}^{n}\sum_{j=0}^{k} f(j,k)$ with $f(j,k)\in\mathbb{Q}(q)$, one can use "Guess" to find such an equation.

Example for Garoufalidis' conjecture

Let $f(n) = (1 - q^n) J_K(n)$. Assume that

$$p_r(q, q^n) f(n+r) + (\cdots) f(n+r-1) + \cdots + p_0(q, q^n) f(n) = 0. \quad (1)$$

- If $p_r(q, q^n) = q^{an+b}$, then we call (1) monic.
- If $p_r(q, q^n) = q^{an+b}$ and $p_0(q, q^n) = q^{cn+d}$, then we call (1) bimonic.

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Example 1 Consider the equation of $(1 - q^n)J_K(n)$ with $K = K_{-1}^{\text{twist}}$:

$$q^{2n+2}(q^{2n+1} - 1)f(n+2) + (\cdots)f(n+1) + q^{2n+2}(q^{2n+3} - 1)f(n) = 0.$$

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Our algorithm yields:

$$q^{2n+4}f(n+3) + (\cdots)f(n+2) + (\cdots)f(n+1) + q^{3n+7}f(n) = 0.$$

Rings of q-difference operators

Let $x = q^n$.

$$\mathbb{Q}(q)[x][\partial] \quad \subset \quad \mathbb{Q}(q,x)[\partial]$$

q-Weyl algebra $\qquad$ q-rational algebra

Assume $L = \ell_r \partial^r + \cdots + \ell_1 \partial + \ell_0 \in \mathbb{Q}(q)[x][\partial]$. Then

$$L \circ f(n) = \ell_r f(n+r) + \cdots + \ell_1 f(n+1) + \ell_0 f(n)$$

- Call L an annihilator of f if $L \circ f = 0$.
- Call $\deg_\partial(L) := r$ the order of L, $\mathrm{lc}_\partial(L) := \ell_r$ the leading coeff
- Let $T \in \mathbb{Q}(q)[x][\partial]$. Call T a left multiple of L if $T = PL$, where $P \in \mathbb{Q}(q,x)[\partial]$.

Rings of q-difference operators

Example 2 Let $g(n) = [n]_q := \frac{1-q^n}{1-q}$. Then

$$(q^n - 1)g(n+1) - (q^{n+1} - 1)g(n) = 0.$$

It is equivalent to

$$[(x-1)\partial - qx + 1] \circ g(n) = 0.$$

Set $P = (x-1)\partial - qx + 1$ and $Q = \frac{1}{qx-1}(\partial - q)$. Then

$$\begin{aligned} T &= QP \\ &= 1\partial^2 - (q+1)\partial + q \end{aligned}$$

is a left multiple of P.

Desingularization

Let $L \in \mathbb{Q}(q)[x][\partial]$ and $p \mid \mathrm{lc}_\partial(L)$.

Assume $T \in \mathbb{Q}(q)[x][\partial]$ and $\sigma(x) = qx$. Call T a *p*-removed operator of L if

- T is a left multiple of L
- $\sigma^{-k}(\mathrm{lc}_\partial(T)) \mid \frac{1}{p}\,\mathrm{lc}_\partial(L)$, where $k = \deg_\partial(T) - \deg_\partial(L)$.

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- $\sigma^{-k}(\mathrm{lc}_\partial(T)) \mid \frac{1}{p}\,\mathrm{lc}_\partial(L)$, where $k = \deg_\partial(T) - \deg_\partial(L)$.

Let T be a p-removed operator of L. Call T a desingularized operator of L if

$$\deg(\mathrm{lc}_\partial(T)) = \min\{\deg(\mathrm{lc}_\partial(Q)) \mid Q \text{ is a p-removed operator}\}$$

Desingularization

Example 2 (continued) Let $P = (x-1)\partial - qx + 1$ and $Q = \frac{1}{qx-1}(\partial - q)$. Then

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is a desingularized operator of P.

Goal: Given $P \in \mathbb{Q}(q)[x][\partial]$, how to compute a desingularized operator of P?

Order bound for desingularized operators

Let $L \in \mathbb{Q}(q)[x][\partial]$.

Assume $p \mid \mathrm{lc}_\partial(L)$, p is irreducible.

- If $p = x$, then p is not removable from L.
- If $p \neq x$ and p is removable, then one can compute an integer k, s.t. there exists a p-removing operator of order k.
- Using Euclidean algorithm, one can compute an order bound for desingularized operators.

Koutschan and Z. Desingularization in the q-Weyl algebra. *Adv. Appl. Math. 97, pp. 80–101, 2018*

Chen et al. Desingularization explains order-degree curves for Ore operators. *ISSAC 2013*.

Determining the k-th submodule

Given $L \in \mathbb{Q}(q)[x][\partial]$, $\deg_\partial(L) = r$.

Set $k \geq r$. Call

$$M_k := \{T \mid T \text{ is a left multiple of } L, \ \deg_\partial(T) \leq k\}$$

the k-th submodule of L.

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Question: Given $k \geq r$, compute a $\mathbb{Q}(q)[x]$-spanning set of M_k?

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Question: Given $k \geq r$, compute a $\mathbb{Q}(q)[x]$-spanning set of M_k?

1. Make an ansatz: $F = z_k\partial^k + \ldots + z_0$,
   where $z_k, \ldots, z_0 \in \mathbb{Q}(q)[x]$ are to be determined.
2. Compute $\text{rrem}(F, L) = 0$. It gives:

   $$(z_k, \ldots, z_0)A = \mathbf{0}, \qquad (2)$$

   where $A \in \mathbb{Q}(q)[x]^{(k+1)\times r}$.
3. Using Gröbner bases or linear algebra, solve (2).

Computing desingularized operators

Let $L \in \mathbb{Q}(q)[x][\partial]$, $\deg_{\partial}(L) = r$.

Question: Assume k is an order bound for desingularized operators of L, compute a desingularized operator?

Computing desingularized operators

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Set $k \geq r$. Call

$$I_k := \left\{ [\partial^k]P \mid P \in M_k \right\} \cup \{0\},$$

the k-th coefficient ideal of L, where $[\partial^k]P$ is the coefficient of ∂^k in P.

Computing desingularized operators

Proposition 1 If $\{B_1, \ldots, B_t\}$ is a spanning set of M_k, then

$$I_k = \langle [\partial^k] B_1, \ldots, [\partial^k] B_t \rangle$$

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Theorem 1 If s is a nonzero element of I_k with minimal degree, then S in M_k with $\mathrm{lc}_\partial(S) = s$ is a desingularized operator.

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Note: Using Euclidean algorithm over $\mathbb{Q}(q)[x]$, one can compute an operator S with $\mathrm{lc}_\partial(S) = s$.

Computing desingularized operators

Algorithm 1: Given $L \in \mathbb{Q}(q)[x][\partial]$, compute a desingularized operator of L.

1. Compute an order bound k for desingularized operators of L.
2. Compute a spanning set of M_k.
3. Using Euclidean algorithm over $\mathbb{Q}(q)[x]$, compute an operator $S \in M_k$ with $\mathrm{lc}_\partial(S) = s$ such that s is a nonzero element of I_k with minimal degree.
4. Output S.

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Example 1 (Continued) Consider the equation of $(1-q^n)J_K(n)$ with $K = K_{-1}^{\text{twist}}$:

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It is equivalent to

$$[q^2x^2(qx^2-1)\partial^2+(\cdots)\partial+q^2x^2(q^3x^2-1)]\circ f(n)=0.$$

Set $L = q^2x^2(qx^2-1)\partial^2+(\cdots)\partial+q^2x^2(q^3x^2-1)$.

Garoufalidis' conjecture

Using **Algorithm 1**, we have

1. An order bound for desingularized operators of L is 3.
2. A spanning set of M_3 over $\mathbb{Q}(q)[x]$ is $\{S, L\}$ with
$$S = q^4x^2\partial^3 + (\cdots)\partial^2 + (\cdots)\partial + q^7x^3.$$
3. By **Theorem 1**, S is a desingularized operator of L.
4. Output $S \circ f(n) = 0$, which is equivalent to
$$q^{2n+4}f(n+3) + (\cdots)f(n+2) + (\cdots)f(n+1) + q^{3n+7}f(n) = 0.$$

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- **Algorithm 1** can be used for desingularization of trailing coeff of L.
- **Algorithm 1** can be used for verification of item 2 of Garoufalidis' conjecture.

Conclusion

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- An algorithm for computing desingularized operators
- Certify special cases of Garouifalids' conjecture

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Thanks!