Mahler Discrete Residues and Summability for Rational Functions

Yi Zhang

Department of Foundational Mathematics
Xi'an Jiaotong-Liverpool University, Suzhou, China

Joint work with Carlos E. Arreche

Linear Mahler equations

Let $\mathbb{K}$ be an algebraically closed field of char 0, x be an indeterminate, and $p \in \mathbb{Z}_{\geq 2}$.

Consider

$$\ell_r(x)y(x^{p^r}) + \ell_{r-1}(x)y(x^{p^{r-1}}) + \cdots + \ell_0(x)y(x) = f(x), \qquad (1)$$

where $\ell_i, f \in \mathbb{K}[x]$ are given, $y(x)$ is unknown. A solution of (1) is called a Mahler function.

(Mahler 1929): study Mahler equations to prove the transcendence of values of some functions.

Fact: the generating series of any p-automatic sequence (such as the Baum–Sweet and the Rudin–Shapiro sequences) is a Mahler function.

Differential Galois theory

Using differential Galois theory, we can determine the differential-algebraic relations between solutions of Mahler equations.

Example (Roques 2018): A Galoisian proof that the generating series of the Baum-Sweet and Rudin-Shapiro sequences are algebraic independent over $\bar{\mathbb{Q}}(x)$.

Goal: Design effective algorithms for computing differential Galois groups of a given linear Mahler equations.

Discrete residues, telescopers, and Galois theory

Endow $\mathbb{K}(x)$ with one of the $\sigma\delta$-field structures:

(S) $\sigma : f(x) \mapsto f(x+1)$ and $\delta = \frac{d}{dx}$;

(Q) $\sigma : f(x) \mapsto f(qx)$ with $q \in \mathbb{K}^\times$ not root of unity and $\delta = x\frac{d}{dx}$.

Discrete residues, telescopers, and Galois theory

Endow $\mathbb{K}(x)$ with one of the $\sigma\delta$-field structures:

(S) $\sigma : f(x) \mapsto f(x+1)$ and $\delta = \frac{d}{dx}$;

(Q) $\sigma : f(x) \mapsto f(qx)$ with $q \in \mathbb{K}^{\times}$ not root of unity and $\delta = x\frac{d}{dx}$.

Let $z_1, \ldots, z_n \in F$, a $\sigma\delta$-extension of $\mathbb{K}(x)$ with $F^{\sigma} = \mathbb{K}$, satisfying

$$\sigma(z_i) = a_i z_i \qquad \text{for some} \quad a_1, \ldots, a_n \in \mathbb{K}(x)^{\times}.$$

Discrete residues, telescopers, and Galois theory

Endow $\mathbb{K}(x)$ with one of the $\sigma\delta$-field structures:

(S) $\sigma : f(x) \mapsto f(x+1)$ and $\delta = \frac{d}{dx}$;

(Q) $\sigma : f(x) \mapsto f(qx)$ with $q \in \mathbb{K}^\times$ not root of unity and $\delta = x\frac{d}{dx}$.

Let $z_1, \ldots, z_n \in F$, a $\sigma\delta$-extension of $\mathbb{K}(x)$ with $F^\sigma = \mathbb{K}$, satisfying

$$\sigma(z_i) = a_i z_i \qquad \text{for some} \quad a_1, \ldots, a_n \in \mathbb{K}(x)^\times.$$

Proposition (Hardouin-Singer 2008) $z_1, \ldots, z_n$ are δ-dependent over $\mathbb{K}(x)$ iff $\exists\ \mathcal{L}_1, \ldots, \mathcal{L}_n \in \mathbb{K}[\delta]$, linear δ-operators with coefficients in $\mathbb{K}$, not all 0, and $g \in \mathbb{K}(x)$:

$$\mathcal{L}_1\left(\frac{\delta(a_1)}{a_1}\right) + \cdots + \mathcal{L}_n\left(\frac{\delta(a_n)}{a_n}\right) = \sigma(g) - g.$$

Discrete residues, telescopers, and Galois theory

Endow $\mathbb{K}(x)$ with one of the $\sigma\delta$-field structures:

(S) $\sigma : f(x) \mapsto f(x+1)$ and $\delta = \frac{d}{dx}$;

(Q) $\sigma : f(x) \mapsto f(qx)$ with $q \in \mathbb{K}^\times$ not root of unity and $\delta = x\frac{d}{dx}$.

Let $z_1, \ldots, z_n \in F$, a $\sigma\delta$-extension of $\mathbb{K}(x)$ with $F^\sigma = \mathbb{K}$, satisfying

$$\sigma(z_i) = a_i z_i \qquad \text{for some} \;\; a_1, \ldots, a_n \in \mathbb{K}(x)^\times.$$

Proposition (Hardouin-Singer 2008) $z_1, \ldots, z_n$ are δ-dependent over $\mathbb{K}(x)$ iff $\exists\ \mathcal{L}_1, \ldots, \mathcal{L}_n \in \mathbb{K}[\delta]$, linear δ-operators with coefficients in $\mathbb{K}$, not all 0, and $g \in \mathbb{K}(x)$:

$$\mathcal{L}_1\left(\frac{\delta(a_1)}{a_1}\right) + \cdots + \mathcal{L}_n\left(\frac{\delta(a_n)}{a_n}\right) = \sigma(g) - g.$$

(Arreche 2017, Arreche-Z. 2022): Using (q-)discrete residues, there exist constants $m_1, \ldots, m_n \in \mathbb{K}$, not all 0, such that

$$m_1\frac{\delta(a_1)}{a_1} + \cdots + m_n\frac{\delta(a_n)}{a_n} = \sigma(g) - g + c$$

for some $g \in \mathbb{K}(x)$ and $c \in \mathbb{K}$ (with $c = 0$ in case (S)).

Motivation

Proposition (Hardouin-Singer 2008) $z_1, \ldots, z_n$ are δ-dependent over $\mathbb{K}(x)$ iff $\exists\ \mathcal{L}_1, \ldots, \mathcal{L}_n \in \mathbb{K}[\delta]$, linear δ-operators with coefficients in $\mathbb{K}$, not all 0, and $g \in \mathbb{K}(x)$:

$$\mathcal{L}_1\left(\frac{\delta(a_1)}{a_1}\right) + \cdots + \mathcal{L}_n\left(\frac{\delta(a_n)}{a_n}\right) = \sigma(g) - g.$$

It also holds for the Mahler case. Question: How to derive the explicit formulae for $\mathcal{L}_1, \ldots, \mathcal{L}_n$?

Motivation

Proposition (Hardouin-Singer 2008) $z_1, \ldots, z_n$ are δ-dependent over $\mathbb{K}(x)$ iff $\exists\ \mathcal{L}_1, \ldots, \mathcal{L}_n \in \mathbb{K}[\delta]$, linear δ-operators with coefficients in $\mathbb{K}$, not all 0, and $g \in \mathbb{K}(x)$:

$$\mathcal{L}_1\left(\frac{\delta(a_1)}{a_1}\right) + \cdots + \mathcal{L}_n\left(\frac{\delta(a_n)}{a_n}\right) = \sigma(g) - g.$$

It also holds for the Mahler case. Question: How to derive the explicit formulae for $\mathcal{L}_1, \ldots, \mathcal{L}_n$?

Idea: Develop the notion of Mahler discrete residues and derive an effective version of Hardouin-Singer's Proposition in the Mahler case.

Continuous residues

Let $\mathbb{K}$ be an algebraically closed field of char 0, and let $f(x) \in \mathbb{K}(x)$. Make the partial fraction decomposition

$$f(x) = r(x) + \sum_{\alpha \in \mathbb{K}} \sum_{k \geq 1} \frac{c_\alpha(k)}{(x-\alpha)^k},$$

where $r(x) \in \mathbb{K}[x]$ and $c_\alpha(k) \in \mathbb{K}$ (almost all 0).

Continuous residues

Let $\mathbb{K}$ be an algebraically closed field of char 0, and let $f(x) \in \mathbb{K}(x)$. Make the partial fraction decomposition

$$f(x) = r(x) + \sum_{\alpha \in \mathbb{K}} \sum_{k \geq 1} \frac{c_\alpha(k)}{(x-\alpha)^k},$$

where $r(x) \in \mathbb{K}[x]$ and $c_\alpha(k) \in \mathbb{K}$ (almost all 0).

Then $f(x)$ is rationally integrable, *i.e.*, there exists $g(x) \in \mathbb{K}(x)$ such that $g'(x) = f(x)$, if and only if the (continuous first-order) *residues*

$$\mathrm{res}(f, \alpha, 1) := c_\alpha(1) = 0 \qquad \text{for every } \alpha \in \mathbb{K}.$$

Continuous residues

Let $\mathbb{K}$ be an algebraically closed field of char 0, and let $f(x) \in \mathbb{K}(x)$. Make the partial fraction decomposition

$$f(x) = r(x) + \sum_{\alpha \in \mathbb{K}} \sum_{k \geq 1} \frac{c_\alpha(k)}{(x-\alpha)^k},$$

where $r(x) \in \mathbb{K}[x]$ and $c_\alpha(k) \in \mathbb{K}$ (almost all 0).

Then $f(x)$ is rationally integrable, *i.e.*, there exists $g(x) \in \mathbb{K}(x)$ such that $g'(x) = f(x)$, if and only if the (continuous first-order) *residues*

$$\operatorname{res}(f, \alpha, 1) := c_\alpha(1) = 0 \qquad \text{for every } \alpha \in \mathbb{K}.$$

Chen and Singer (2012) created a notion of *discrete residues* that plays an analogous role (where *integrability* $\mapsto$ *summability*) for the *shift* ($x \mapsto x+1$) and *q-dilation* ($x \mapsto qx$) difference operators.

Discrete residues: shift case

Rewrite the partial fraction decomposition of $f(x) \in \mathbb{K}(x)$:

$$f(x) = r(x) + \sum_{k \geq 1} \sum_{[\alpha] \in \mathbb{K}/\mathbb{Z}} \sum_{n \in \mathbb{Z}} \frac{c_\alpha(k, n)}{(x - \alpha + n)^k}$$

where $r(x) \in \mathbb{K}[x]$, $\alpha \in \mathbb{K}$ is a coset representative for $[\alpha] := \alpha + \mathbb{Z} \in \mathbb{K}/\mathbb{Z}$, and $c_\alpha(k, n) \in \mathbb{K}$.

Discrete residues: shift case

Rewrite the partial fraction decomposition of $f(x) \in \mathbb{K}(x)$:

$$f(x) = r(x) + \sum_{k \geq 1} \sum_{[\alpha] \in \mathbb{K}/\mathbb{Z}} \sum_{n \in \mathbb{Z}} \frac{c_\alpha(k,n)}{(x - \alpha + n)^k}$$

where $r(x) \in \mathbb{K}[x]$, $\alpha \in \mathbb{K}$ is a coset representative for $[\alpha] := \alpha + \mathbb{Z} \in \mathbb{K}/\mathbb{Z}$, and $c_\alpha(k,n) \in \mathbb{K}$.

The discrete residue of $f(x) \in \mathbb{K}(x)$ at the $\mathbb{Z}$-orbit $[\alpha] \in \mathbb{K}/\mathbb{Z}$ of order k is defined as

$$\mathrm{dres}(f, [\alpha], k) := \sum_{n \in \mathbb{Z}} c_\alpha(k,n).$$

Discrete residues: shift case

Rewrite the partial fraction decomposition of $f(x) \in \mathbb{K}(x)$:

$$f(x) = r(x) + \sum_{k \geq 1} \sum_{[\alpha] \in \mathbb{K}/\mathbb{Z}} \sum_{n \in \mathbb{Z}} \frac{c_\alpha(k,n)}{(x - \alpha + n)^k}$$

where $r(x) \in \mathbb{K}[x]$, $\alpha \in \mathbb{K}$ is a coset representative for $[\alpha] := \alpha + \mathbb{Z} \in \mathbb{K}/\mathbb{Z}$, and $c_\alpha(k,n) \in \mathbb{K}$.

The discrete residue of $f(x) \in \mathbb{K}(x)$ at the $\mathbb{Z}$-orbit $[\alpha] \in \mathbb{K}/\mathbb{Z}$ of order k is defined as

$$\mathrm{dres}(f, [\alpha], k) := \sum_{n \in \mathbb{Z}} c_\alpha(k,n).$$

Proposition (Chen-Singer 2012) $f(x)$ is rationally summable, *i.e.*, there exists $g(x) \in \mathbb{K}(x)$ such that $f(x) = g(x+1) - g(x)$, if and only if $\mathrm{dres}(f, [\alpha], k) = 0$ for each $[\alpha] \in \mathbb{K}/\mathbb{Z}$ and $k \in \mathbb{N}$.

Discrete residues: q-dilation case

Fix $q \in \mathbb{K}^\times$ not a root of unity, and write $f(x) \in \mathbb{K}(x)$:

$$f(x) = r(x) + \sum_{k \geq 1} \sum_{[\alpha]_q \in \mathbb{K}^\times / q^{\mathbb{Z}}} \sum_{n \in \mathbb{Z}} \frac{c_\alpha(k,n)}{(x - q^n \alpha)^k}$$

where $r(x) = \sum_{j \in \mathbb{Z}} r_j x^j \in \mathbb{K}[x, x^{-1}]$ and $\alpha \in \mathbb{K}^\times$ is a coset representative for $[\alpha]_q := \alpha \cdot q^{\mathbb{Z}} \in \mathbb{K}^\times / q^{\mathbb{Z}}$.

Discrete residues: q-dilation case

Fix $q \in \mathbb{K}^\times$ not a root of unity, and write $f(x) \in \mathbb{K}(x)$:

$$f(x) = r(x) + \sum_{k \geq 1} \sum_{[\alpha]_q \in \mathbb{K}^\times / q^{\mathbb{Z}}} \sum_{n \in \mathbb{Z}} \frac{c_\alpha(k, n)}{(x - q^n \alpha)^k}$$

where $r(x) = \sum_{j \in \mathbb{Z}} r_j x^j \in \mathbb{K}[x, x^{-1}]$ and $\alpha \in \mathbb{K}^\times$ is a coset representative for $[\alpha]_q := \alpha \cdot q^{\mathbb{Z}} \in \mathbb{K}^\times / q^{\mathbb{Z}}$.

The *q-discrete residue* of $f(x)$ at the $q^{\mathbb{Z}}$-orbit $[\alpha]_q \in \mathbb{K}^\times / q^{\mathbb{Z}}$ of order k (resp., at infinity) is defined as

$$q\text{-}\mathrm{dres}(f, [\alpha]_q, k) := \sum_{n \in \mathbb{Z}} q^{-kn} c_\alpha(k, n) \quad (\text{resp., } \ q\text{-}\mathrm{dres}(f, \infty) := r_0).$$

Discrete residues: q-dilation case

Fix $q \in \mathbb{K}^\times$ not a root of unity, and write $f(x) \in \mathbb{K}(x)$:

$$f(x) = r(x) + \sum_{k \geq 1} \sum_{[\alpha]_q \in \mathbb{K}^\times/q^{\mathbb{Z}}} \sum_{n \in \mathbb{Z}} \frac{c_\alpha(k,n)}{(x - q^n\alpha)^k}$$

where $r(x) = \sum_{j \in \mathbb{Z}} r_j x^j \in \mathbb{K}[x, x^{-1}]$ and $\alpha \in \mathbb{K}^\times$ is a coset representative for $[\alpha]_q := \alpha \cdot q^{\mathbb{Z}} \in \mathbb{K}^\times/q^{\mathbb{Z}}$.

The q-discrete residue of $f(x)$ at the $q^{\mathbb{Z}}$-orbit $[\alpha]_q \in \mathbb{K}^\times/q^{\mathbb{Z}}$ of order k (resp., at infinity) is defined as

$$q\text{-dres}(f, [\alpha]_q, k) := \sum_{n \in \mathbb{Z}} q^{-kn} c_\alpha(k,n) \quad (\text{resp., } q\text{-dres}(f, \infty) := r_0).$$

Proposition (Chen-Singer 2012) $f(x)$ is rationally q-summable, *i.e.*, there exists $g(x) \in \mathbb{K}(x)$ such that $f(x) = g(qx) - g(x)$, if and only if $q\text{-dres}(f, \infty) = 0$ and $q\text{-dres}(f, [\alpha]_q, k) = 0$ for each $[\alpha]_q \in \mathbb{K}^\times/q^{\mathbb{Z}}$ and $k \in \mathbb{N}$.

Why use residues?

An advantage of using residues is to answer *whether* (yes/no) $f(x) \in \mathbb{K}(x)$ is

- rationally integrable: $f(x) = g'(x)$; or
- rationally summable: $f(x) = g(x+1) - g(x)$; or
- rationally q-summable: $f(x) = g(qx) - g(x)$;

without computing the certificate $g(x) \in \mathbb{K}(x)$.

Why use residues?

An advantage of using residues is to answer *whether* (yes/no) $f(x) \in \mathbb{K}(x)$ is

- rationally integrable: $f(x) = g'(x)$; or
- rationally summable: $f(x) = g(x+1) - g(x)$; or
- rationally q-summable: $f(x) = g(qx) - g(x)$;

without computing the certificate $g(x) \in \mathbb{K}(x)$.

Computing residues from the definition is impractical because it requires doing partial fraction decompositions.

Why use residues?

An advantage of using residues is to answer *whether* (yes/no) $f(x) \in \mathbb{K}(x)$ is

- rationally integrable: $f(x) = g'(x)$; or
- rationally summable: $f(x) = g(x+1) - g(x)$; or
- rationally q-summable: $f(x) = g(qx) - g(x)$;

without computing the certificate $g(x) \in \mathbb{K}(x)$.

Computing residues from the definition is impractical because it requires doing partial fraction decompositions.

In the differential case, there is a better way: if $f = \frac{a}{b}$ with $a, b \in \mathbb{K}[x]$, $\gcd(a, b) = 1$, $\deg(a) < \deg(b)$, and b squarefree, then the roots of the Rothstein-Trager resultant

$$RT(f) := \mathrm{Res}_x(a - z \cdot b', b) \in \mathbb{K}[z]$$

are precisely the first-order continuous residues of $f(x)$, which implies $f(x)$ is rationally integrable iff $RT(f)$ is a monomial in z.

Mahler summability for rational functions

Fix $p \in \mathbb{Z}_{\geq 2}$ and let the Mahler difference operator $\sigma : g(x) \mapsto g(x^p)$ for $g(x) \in \mathbb{K}(x)$.

Mahler summability for rational functions

Fix $p \in \mathbb{Z}_{\geq 2}$ and let the Mahler difference operator $\sigma : g(x) \mapsto g(x^p)$ for $g(x) \in \mathbb{K}(x)$.

We say $f(x) \in \mathbb{K}(x)$ is Mahler summable if there exists $g(x) \in \mathbb{K}(x)$ such that $f(x) = g(x^p) - g(x)$.

Mahler summability for rational functions

Fix $p \in \mathbb{Z}_{\geq 2}$ and let the Mahler difference operator $\sigma : g(x) \mapsto g(x^p)$ for $g(x) \in \mathbb{K}(x)$.

We say $f(x) \in \mathbb{K}(x)$ is Mahler summable if there exists $g(x) \in \mathbb{K}(x)$ such that $f(x) = g(x^p) - g(x)$.

Mahler Summability Problem: given $f(x) \in \mathbb{K}(x)$, decide effectively whether $f(x)$ is Mahler summable.

- Done by Chyzak-Dreyfus-Dumas-Mezzarobba (2018).

Mahler summability for rational functions

Fix $p \in \mathbb{Z}_{\geq 2}$ and let the Mahler difference operator $\sigma : g(x) \mapsto g(x^p)$ for $g(x) \in \mathbb{K}(x)$.

We say $f(x) \in \mathbb{K}(x)$ is Mahler summable if there exists $g(x) \in \mathbb{K}(x)$ such that $f(x) = g(x^p) - g(x)$.

Mahler Summability Problem: given $f(x) \in \mathbb{K}(x)$, decide effectively whether $f(x)$ is Mahler summable.

- Done by Chyzak-Dreyfus-Dumas-Mezzarobba (2018).

Our Goal: Construct a ($\mathbb{K}$-linear) complete obstruction to the Mahler summability of $f(x) \in \mathbb{K}(x)$.

Mahler summability for rational functions

More precisely, for the $\mathbb{K}$-linear map $\Delta : g(x) \mapsto g(x^p) - g(x)$, we wish to construct explicitly a $\mathbb{K}$-linear map ∇ on $\mathbb{K}(x)$ such that $\mathrm{im}(\Delta) = \ker(\nabla)$, bypassing computation of certificates.

We call ∇ the Mahler reduction operator. Given $f \in \mathbb{K}(x)$, set $\bar{f} = \nabla(f)$. Then f is Mahler summable if and only if $\bar{f} = 0$. The numerators in the partial fraction decomposition of $\bar{f}$ are Mahler discrete residues of f.

Mahler trajectories and Mahler trees

Let $\mathcal{P} = \{p^n \mid n \in \mathbb{Z}_{\geq 0}\}$ denote the multiplicative monoid of non-negative powers of p.

Mahler trajectories and Mahler trees

Let $\mathcal{P} = \{p^n \mid n \in \mathbb{Z}_{\geq 0}\}$ denote the multiplicative monoid of non-negative powers of p.

We denote by $\mathbb{Z}/\mathcal{P}$ the set of maximal trajectories for the action of $\mathcal{P}$ on $\mathbb{Z}$ by multiplication:

$$\mathbb{Z}/\mathcal{P} = \big\{\{0\}\big\} \cup \big\{\{ip^n \mid n \in \mathbb{Z}_{\geq 0}\} \bigm| i \in \mathbb{Z} \text{ such that } p \nmid i\big\}.$$

The elements $\theta \in \mathbb{Z}/\mathcal{P}$ are pairwise disjoint subsets of $\mathbb{Z}$ whose union is all of $\mathbb{Z}$.

Mahler trajectories and Mahler trees

Let $\mathcal{P} = \{p^n \mid n \in \mathbb{Z}_{\geq 0}\}$ denote the multiplicative monoid of non-negative powers of p.

We denote by $\mathbb{Z}/\mathcal{P}$ the set of maximal trajectories for the action of $\mathcal{P}$ on $\mathbb{Z}$ by multiplication:

$$\mathbb{Z}/\mathcal{P} = \big\{\{0\}\big\} \cup \Big\{\{ip^n \mid n \in \mathbb{Z}_{\geq 0}\} \;\Big|\; i \in \mathbb{Z} \text{ such that } p \nmid i\Big\}.$$

The elements $\theta \in \mathbb{Z}/\mathcal{P}$ are pairwise disjoint subsets of $\mathbb{Z}$ whose union is all of $\mathbb{Z}$.

We denote by $\mathcal{T}_M$ the set of equivalence classes for the equivalence relation on $\mathbb{K}^\times$ defined by $\alpha \sim \gamma$ if and only if $\alpha^{p^s} = \gamma^{p^r}$ for some $r, s \in \mathbb{Z}_{\geq 0}$.

The elements $\tau \in \mathcal{T}_M$, called Mahler trees, are pairwise disjoint subsets of $\mathbb{K}^\times$ whose union is all of $\mathbb{K}^\times$. We write $\tau(\alpha)$ for the unique Mahler tree containing $\alpha \in \mathbb{K}^\times$.

Examples of Mahler trees

We define a digraph on the vertex set τ for each Mahler tree $\tau \in \mathcal{T}_M$ with one directed edge $\alpha \to \gamma$ whenever $\alpha^p = \gamma$.

Examples of Mahler trees

We define a digraph on the vertex set τ for each Mahler tree $\tau \in \mathcal{T}_M$ with one directed edge $\alpha \to \gamma$ whenever $\alpha^p = \gamma$.

With $p = 3$, the vertices of $\tau(2)$ near $2 \in \mathbb{K}^\times$ are

$$\begin{tikzcd}
\left\{\sqrt[9]{2}, \zeta_3\sqrt[9]{2}, \zeta_3^2\sqrt[9]{2}\right\} \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \sqrt[3]{2} \arrow[dr] & & & \\
\left\{\zeta_9\sqrt[9]{2}, \zeta_9^4\sqrt[9]{2}, \zeta_9^7\sqrt[9]{2}\right\} \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \zeta_3\sqrt[3]{2} \arrow[r] & 2 \arrow[r] & 8 \arrow[r] & 512 \\
\left\{\zeta_9^2\sqrt[9]{2}, \zeta_9^5\sqrt[9]{2}, \zeta_9^8\sqrt[9]{2}\right\} \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \zeta_3^2\sqrt[3]{2} \arrow[ur] & & &
\end{tikzcd}$$

Examples of Mahler trees

We define a digraph on the vertex set τ for each Mahler tree $\tau \in \mathcal{T}_M$ with one directed edge $\alpha \to \gamma$ whenever $\alpha^p = \gamma$.

With $p = 3$, the vertices of $\tau(2)$ near $2 \in \mathbb{K}^\times$ are

$$
\begin{tikzcd}
\left\{\sqrt[9]{2}, \zeta_3 \sqrt[9]{2}, \zeta_3^2 \sqrt[9]{2}\right\} \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \sqrt[3]{2} \arrow[dr] & & & \\
\left\{\zeta_9 \sqrt[9]{2}, \zeta_9^4 \sqrt[9]{2}, \zeta_9^7 \sqrt[9]{2}\right\} \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \zeta_3 \sqrt[3]{2} \arrow[r] & 2 \arrow[r] & 8 \arrow[r] & 512 \\
\left\{\zeta_9^2 \sqrt[9]{2}, \zeta_9^5 \sqrt[9]{2}, \zeta_9^8 \sqrt[9]{2}\right\} \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \zeta_3^2 \sqrt[3]{2} \arrow[ur] & & &
\end{tikzcd}
$$

With $p = 3$, the vertices of $\tau(\zeta_4)$ near $\zeta_4 \in \mathbb{K}^\times$ are

$$
\begin{tikzcd}
\left\{\zeta_{36}, \zeta_{36}^{13}, \zeta_{36}^{25}\right\} \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] & & & \left\{\zeta_{36}^{7}, \zeta_{36}^{19}, \zeta_{36}^{31}\right\} \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] \\
\zeta_{12} \arrow[dr] & & & \zeta_{12}^{7} \arrow[dl] \\
& \zeta_4 \arrow[r, bend left] & \zeta_4^3 \arrow[l, bend left] & \\
\zeta_{12}^{5} \arrow[ur] & & & \zeta_{12}^{11} \arrow[ul] \\
\left\{\zeta_{36}^{5}, \zeta_{36}^{17}, \zeta_{36}^{29}\right\} \arrow[u, shift left=2] \arrow[u] \arrow[u, shift right=2] & & & \left\{\zeta_{36}^{11}, \zeta_{36}^{23}, \zeta_{36}^{35}\right\} \arrow[u, shift left=2] \arrow[u] \arrow[u, shift right=2]
\end{tikzcd}
$$

Mahler decomposition of partial fractions

For $f(x) \in \mathbb{K}(x)$, we can decompose it uniquely as $f(x) = f_L(x) + f_T(x)$:

$$f_L(x) := \sum_{j \in \mathbb{Z}} r_j x^j \qquad \text{and} \qquad f_T(x) := \sum_{k \geq 1} \sum_{\alpha \in \mathbb{K}^\times} \frac{c_\alpha(k)}{(x-\alpha)^k},$$

where $r_j, c_\alpha(k) \in \mathbb{K}$.

Mahler decomposition of partial fractions

For $f(x) \in \mathbb{K}(x)$, we can decompose it uniquely as $f(x) = f_L(x) + f_T(x)$:

$$f_L(x) := \sum_{j \in \mathbb{Z}} r_j x^j \qquad \text{and} \qquad f_T(x) := \sum_{k \geq 1} \sum_{\alpha \in \mathbb{K}^\times} \frac{c_\alpha(k)}{(x-\alpha)^k},$$

where $r_j, c_\alpha(k) \in \mathbb{K}$.

This yields a σ-stable $\mathbb{K}$-vector space decomposition of $\mathbb{K}(x)$ as

$$\mathbb{K}(x) \simeq \mathbb{K}[x, x^{-1}] \oplus \mathbb{K}(x)_T.$$

Mahler decomposition of partial fractions

For $f(x) \in \mathbb{K}(x)$, we can decompose it uniquely as $f(x) = f_L(x) + f_T(x)$:

$$f_L(x) := \sum_{j \in \mathbb{Z}} r_j x^j \qquad \text{and} \qquad f_T(x) := \sum_{k \geq 1} \sum_{\alpha \in \mathbb{K}^\times} \frac{c_\alpha(k)}{(x-\alpha)^k},$$

where $r_j, c_\alpha(k) \in \mathbb{K}$.

This yields a σ-stable $\mathbb{K}$-vector space decomposition of $\mathbb{K}(x)$ as

$$\mathbb{K}(x) \simeq \mathbb{K}[x, x^{-1}] \oplus \mathbb{K}(x)_T.$$

Then $f \in \mathbb{K}(x)$ is Mahler summable if and only if f_L and f_T are both Mahler summable. We address each component separately.

Mahler decomposition of partial fractions

For $f(x) \in \mathbb{K}(x)$, we can decompose it uniquely as $f(x) = f_L(x) + f_T(x)$:

$$f_L(x) := \sum_{j\in\mathbb{Z}} r_j x^j \qquad \text{and} \qquad f_T(x) := \sum_{k\geq 1} \sum_{\alpha \in \mathbb{K}^\times} \frac{c_\alpha(k)}{(x-\alpha)^k},$$

where $r_j, c_\alpha(k) \in \mathbb{K}$.

This yields a σ-stable $\mathbb{K}$-vector space decomposition of $\mathbb{K}(x)$ as

$$\mathbb{K}(x) \simeq \mathbb{K}[x, x^{-1}] \oplus \mathbb{K}(x)_T.$$

Then $f \in \mathbb{K}(x)$ is Mahler summable if and only if f_L and f_T are both Mahler summable. We address each component separately.

Moreover, the decompositions $f_L = \sum_{\theta\in\mathbb{Z}/\mathcal{P}} f_\theta$ and $f_T = \sum_{\tau\in\mathcal{T}_M} f_\tau$:

$$f_\theta := \sum_{j\in\theta} r_j x^j \qquad \text{and} \qquad f_\tau := \sum_{k\geq 1} \sum_{\alpha \in \tau} \frac{c_\alpha(k)}{(x-\alpha)^k}$$

are also σ-stable. Can decide summability of f by deciding for each f_θ ($\theta \in \mathbb{Z}/\mathcal{P}$) and each f_τ ($\tau \in \mathcal{T}_M$) individually.

Mahler residues at infinity

Definition (Arreche-Z. 2022) Let $f(x) \in \mathbb{K}(x)$ and write $f_L(x) = \sum_{j\in\mathbb{Z}} r_j x^j \in \mathbb{K}[x, x^{-1}]$. The Mahler residue of $f(x)$ at infinity is the vector

$$\operatorname{dres}(f, \infty) := \left(\sum_{j\in\theta} r_j \right)_{\theta\in\mathbb{Z}/\mathcal{P}} \in \bigoplus_{\theta\in\mathbb{Z}/\mathcal{P}} \mathbb{K}.$$

Mahler residues at infinity

Definition (Arreche-Z. 2022) Let $f(x) \in \mathbb{K}(x)$ and write $f_L(x) = \sum_{j\in\mathbb{Z}} r_j x^j \in \mathbb{K}[x, x^{-1}]$. The Mahler residue of $f(x)$ at infinity is the vector

$$\mathrm{dres}(f, \infty) := \left(\sum_{j\in\theta} r_j\right)_{\theta\in\mathbb{Z}/\mathcal{P}} \in \bigoplus_{\theta\in\mathbb{Z}/\mathcal{P}} \mathbb{K}.$$

Proposition (Arreche-Z. 2022) For $f(x) \in \mathbb{K}(x)$ the component $f_L(x) \in \mathbb{K}[x, x^{-1}]$ is Mahler summable if and only if $\mathrm{dres}(f, \infty) = 0$ (the zero vector).

Mahler residues at infinity

Definition (Arreche-Z. 2022) Let $f(x) \in \mathbb{K}(x)$ and write $f_L(x) = \sum_{j\in\mathbb{Z}} r_j x^j \in \mathbb{K}[x, x^{-1}]$. The Mahler residue of $f(x)$ at infinity is the vector

$$\mathrm{dres}(f, \infty) := \left(\sum_{j\in\theta} r_j \right)_{\theta \in \mathbb{Z}/\mathcal{P}} \in \bigoplus_{\theta \in \mathbb{Z}/\mathcal{P}} \mathbb{K}.$$

Proposition (Arreche-Z. 2022) For $f(x) \in \mathbb{K}(x)$ the component $f_L(x) \in \mathbb{K}[x, x^{-1}]$ is Mahler summable if and only if $\mathrm{dres}(f, \infty) = 0$ (the zero vector).

Proof sketch: For $\theta = \{ip^n\}$ with $p \nmid i$, let $\bar{f}_\theta^{(n)} = f_\theta + \Delta(g_\theta^{(n)})$ with $g_\theta^{(0)} := 0$ and $g_\theta^{(n+1)} := g_\theta^{(n)} + \left(\sum_{\ell=0}^{n} r_{ip^\ell}\right) x^{ip^n}$. Then, for h largest s.t. $r_{ip^h} \neq 0$, $\bar{f}^{(h)} = \mathrm{dres}(f, \infty)_\theta \cdot x^{ip^h}$. A *dispersion* argument shows $\bar{f}_\theta^{(h)} = 0$ iff f_θ is Mahler summable.

Mahler residues at Mahler trees (1 of 3): coefficients

For $\alpha \in \mathbb{K}^\times$, ζ_p a primitive p-th root of unity, let $V_k^m(\zeta_p^i \alpha) \in \mathbb{K}$:

$$\sigma\left(\frac{1}{(x-\alpha^p)^m}\right) = \frac{1}{(x^p-\alpha^p)^m} = \sum_{k=1}^{m}\sum_{i=0}^{p-1}\frac{V_k^m(\zeta_p^i\alpha)}{(x-\zeta_p^i\alpha)^k}.$$

Mahler residues at Mahler trees (1 of 3): coefficients

For $\alpha \in \mathbb{K}^\times$, ζ_p a primitive p-th root of unity, let $V_k^m(\zeta_p^i\alpha) \in \mathbb{K}$:

$$\sigma\left(\frac{1}{(x-\alpha^p)^m}\right) = \frac{1}{(x^p-\alpha^p)^m} = \sum_{k=1}^{m}\sum_{i=0}^{p-1}\frac{V_k^m(\zeta_p^i\alpha)}{(x-\zeta_p^i\alpha)^k}.$$

The definition of Mahler residues is more complicated because applying $\sigma : f(x) \mapsto f(x^p)$ at poles of order m "leaks" into the poles of order $k \leq m$.

Mahler residues at Mahler trees (1 of 3): coefficients

For $\alpha \in \mathbb{K}^\times$, ζ_p a primitive p-th root of unity, let $V_k^m(\zeta_p^i\alpha) \in \mathbb{K}$:

$$\sigma\left(\frac{1}{(x-\alpha^p)^m}\right) = \frac{1}{(x^p-\alpha^p)^m} = \sum_{k=1}^{m}\sum_{i=0}^{p-1}\frac{V_k^m(\zeta_p^i\alpha)}{(x-\zeta_p^i\alpha)^k}.$$

The definition of Mahler residues is more complicated because applying $\sigma : f(x) \mapsto f(x^p)$ at poles of order m "leaks" into the poles of order $k \leq m$.

Lemma (Arreche-Z. 2022)

$$V_k^m(\zeta_p^i\alpha) = \mathbb{V}_k^m \cdot \frac{(\zeta_p^i\alpha)^k}{\alpha^{pm}},$$

where $\mathbb{V}_k^m \in \mathbb{Q}$ are obtained from the Taylor coefficients at $x = 1$:

$$(x^{p-1}+\cdots+x+1)^{-m} = \sum_{k=1}^{m}\mathbb{V}_k^m \cdot (x-1)^{m-k} + O((x-1)^m).$$

Mahler residues at Mahler trees (1 of 3): coefficients

For $\alpha \in \mathbb{K}^\times$, ζ_p a primitive p-th root of unity, let $V_k^m(\zeta_p^i\alpha) \in \mathbb{K}$:

$$\sigma\left(\frac{1}{(x-\alpha^p)^m}\right) = \frac{1}{(x^p-\alpha^p)^m} = \sum_{k=1}^{m}\sum_{i=0}^{p-1}\frac{V_k^m(\zeta_p^i\alpha)}{(x-\zeta_p^i\alpha)^k}.$$

The definition of Mahler residues is more complicated because applying $\sigma : f(x) \mapsto f(x^p)$ at poles of order m "leaks" into the poles of order $k \leq m$.

Lemma (Arreche-Z. 2022)

$$V_k^m(\zeta_p^i\alpha) = \mathbb{V}_k^m \cdot \frac{(\zeta_p^i\alpha)^k}{\alpha^{pm}},$$

where $\mathbb{V}_k^m \in \mathbb{Q}$ are obtained from the Taylor coefficients at $x = 1$:

$$(x^{p-1} + \cdots + x + 1)^{-m} = \sum_{k=1}^{m}\mathbb{V}_k^m \cdot (x-1)^{m-k} + O((x-1)^m).$$

The "universal coefficients" $\mathbb{V}_k^m$ can be computed directly (as a sum over partitions) using the Faà di Bruno's formula.

Small example of Mahler coefficients

Let $p = 3$, $m = 2$, and $\alpha^3 = 1$. Then

$$\sigma\left(\frac{1}{(x-1)^2}\right) = \frac{1}{(x^3-1)^2} = \sum_{k=1}^{2}\sum_{i=0}^{2}\frac{V_k^2(\zeta_3^i)}{(x-\zeta_3^i)^k},$$

By the previous Lemma, $V_k^2(\zeta_3^i) = \mathbb{V}_k^2 \cdot (\zeta_3^i)^{k-6} = \mathbb{V}_k^2 \cdot \zeta_3^{ki}$ for $k = 1, 2$. We find that

$$\mathbb{V}_2^2 = \left(x^2+x+1\right)^{-2}\Big|_{x=1} = \tfrac{1}{9};\ \text{ and }\ \mathbb{V}_1^2 = \left((x^2+x+1)^{-2}\right)'\Big|_{x=1} = -\tfrac{2}{9}.$$

Using a computer algebra system (or by hand!), one can very that the partial fraction decomposition of $9 \cdot (x^3-1)^{-2}$ is indeed

$$\frac{1}{(x-1)^2} + \frac{\zeta_3^2}{(x-\zeta_3)^2} + \frac{\zeta_3}{(x-\zeta_3^2)^2} + \frac{-2}{x-1} + \frac{-2\zeta_3}{x-\zeta_3} + \frac{-2\zeta_3^2}{x-\zeta_3^2}.$$

Mahler residues at Mahler trees (2 of 3): definition

Suppose $\gamma \in \mathbb{K}^\times$ is not a root of unity, $f \in \mathbb{K}(x)$, and $h \in \mathbb{Z}_{\geq 0}$ s.t.:

$$\mathrm{sing}(f) \cap \tau(\gamma) \subseteq \{\zeta_{p^n}^i \gamma^{p^{h-n}} \mid 0 \leq n \leq h,\ i \in \mathbb{Z}/p^n\mathbb{Z}\}.$$

Mahler residues at Mahler trees (2 of 3): definition

Suppose $\gamma \in \mathbb{K}^\times$ is not a root of unity, $f \in \mathbb{K}(x)$, and $h \in \mathbb{Z}_{\geq 0}$ s.t.:

$$\mathrm{sing}(f) \cap \tau(\gamma) \subseteq \{\zeta_{p^n}^i \gamma^{p^{h-n}} \mid 0 \leq n \leq h,\ i \in \mathbb{Z}/p^n\mathbb{Z}\}.$$

Then we can write, for $\tau = \tau(\gamma)$,

$$f_\tau = \sum_{k=1}^{m} \sum_{n=0}^{h} \sum_{i \in \mathbb{Z}/p^n\mathbb{Z}} \frac{c_\gamma(k,n,i)}{(x - \zeta_{p^n}^i \gamma^{p^{h-n}})^k}.$$

Mahler residues at Mahler trees (2 of 3): definition

Suppose $\gamma \in \mathbb{K}^\times$ is not a root of unity, $f \in \mathbb{K}(x)$, and $h \in \mathbb{Z}_{\geq 0}$ s.t.:

$$\mathrm{sing}(f) \cap \tau(\gamma) \subseteq \{\zeta_{p^n}^i \gamma^{p^{h-n}} \mid 0 \leq n \leq h,\ i \in \mathbb{Z}/p^n\mathbb{Z}\}.$$

Then we can write, for $\tau = \tau(\gamma)$,

$$f_\tau = \sum_{k=1}^{m} \sum_{n=0}^{h} \sum_{i \in \mathbb{Z}/p^n\mathbb{Z}} \frac{c_\gamma(k,n,i)}{(x - \zeta_{p^n}^i \gamma^{p^{h-n}})^k}.$$

Set recursively: $\tilde{c}_{k,0,0} := c_\gamma(k,0,0)$, and for $1 \leq n \leq h$; $i \in \mathbb{Z}/p^n\mathbb{Z}$:

$$\tilde{c}_{k,n,i} := c_\gamma(k,n,i) + \sum_{j=k}^{m} \tilde{c}_{j,n-1,\pi_{n-1}(i)} V_k^j(\zeta_{p^n}^i \gamma^{p^{h-n}}),$$

where $\pi_{n-1} : \mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}/p^{n-1}\mathbb{Z}$ is the canonical projection.

Mahler residues at Mahler trees (2 of 3): definition

Suppose $\gamma \in \mathbb{K}^\times$ is not a root of unity, $f \in \mathbb{K}(x)$, and $h \in \mathbb{Z}_{\geq 0}$ s.t.:

$$\operatorname{sing}(f) \cap \tau(\gamma) \subseteq \{\zeta_{p^n}^i \gamma^{p^{h-n}} \mid 0 \leq n \leq h,\ i \in \mathbb{Z}/p^n\mathbb{Z}\}.$$

Then we can write, for $\tau = \tau(\gamma)$,

$$f_\tau = \sum_{k=1}^{m} \sum_{n=0}^{h} \sum_{i \in \mathbb{Z}/p^n\mathbb{Z}} \frac{c_\gamma(k,n,i)}{(x - \zeta_{p^n}^i \gamma^{p^{h-n}})^k}.$$

Set recursively: $\tilde{c}_{k,0,0} := c_\gamma(k,0,0)$, and for $1 \leq n \leq h$; $i \in \mathbb{Z}/p^n\mathbb{Z}$:

$$\tilde{c}_{k,n,i} := c_\gamma(k,n,i) + \sum_{j=k}^{m} \tilde{c}_{j,n-1,\pi_{n-1}(i)} V_k^j(\zeta_{p^n}^i \gamma^{p^{h-n}}),$$

where $\pi_{n-1} : \mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}/p^{n-1}\mathbb{Z}$ is the canonical projection.

Mahler residues at Mahler trees (2 of 3): definition

Suppose $\gamma \in \mathbb{K}^\times$ is not a root of unity, $f \in \mathbb{K}(x)$, and $h \in \mathbb{Z}_{\geq 0}$ s.t.:

$$\mathrm{sing}(f) \cap \tau(\gamma) \subseteq \{\zeta_{p^n}^i \gamma^{p^{h-n}} \mid 0 \leq n \leq h,\ i \in \mathbb{Z}/p^n\mathbb{Z}\}.$$

Then we can write, for $\tau = \tau(\gamma)$,

$$f_\tau = \sum_{k=1}^{m} \sum_{n=0}^{h} \sum_{i \in \mathbb{Z}/p^n\mathbb{Z}} \frac{c_\gamma(k,n,i)}{(x - \zeta_{p^n}^i \gamma^{p^{h-n}})^k}.$$

Set recursively: $\tilde{c}_{k,0,0} := c_\gamma(k,0,0)$, and for $1 \leq n \leq h$; $i \in \mathbb{Z}/p^n\mathbb{Z}$:

$$\tilde{c}_{k,n,i} := c_\gamma(k,n,i) + \sum_{j=k}^{m} \tilde{c}_{j,n-1,\pi_{n-1}(i)} V_k^j(\zeta_{p^n}^i \gamma^{p^{h-n}}),$$

where $\pi_{n-1} : \mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}/p^{n-1}\mathbb{Z}$ is the canonical projection.
Definition (Arreche-Z. 2022) The Mahler discrete residue at τ of order k is the vector $\mathrm{dres}(f,\tau,k) \in \bigoplus_{\alpha \in \tau} \mathbb{K}$ with α-component $:= 0$ except possibly at $\alpha = \zeta_{p^h}^i \gamma$ for $i \in \mathbb{Z}/p^h\mathbb{Z}$, given by $\tilde{c}_{k,h,i}$.

Mahler residues at Mahler trees (3 of 3): proof

Proposition (Arreche-Z. 2022) For $f \in \mathbb{K}(x)$ the component f_T is Mahler summable if and only if $\mathrm{dres}(f, \tau, k) = 0$ (the zero vector) for each $\tau \in \mathcal{T}_M$ and $k \in \mathbb{N}$.

Mahler residues at Mahler trees (3 of 3): proof

Proposition (Arreche-Z. 2022) For $f \in \mathbb{K}(x)$ the component f_T is Mahler summable if and only if $\mathrm{dres}(f, \tau, k) = 0$ (the zero vector) for each $\tau \in \mathcal{T}_M$ and $k \in \mathbb{N}$.

Proof idea. Similar to the Laurent polynomial case, one adds to f_τ a sequence of "small" summable elements until one obtains a "remainder"

$$\bar{f}_\tau = \sum_{k \geq 1} \sum_{\alpha \in \tau} \frac{\mathrm{dres}(f, \tau, k)_\alpha}{(x - \alpha)^k}$$

such that f_τ is Mahler summable iff $\bar{f}_\tau = 0$.

Mahler residues at Mahler trees (3 of 3): proof

Proposition (Arreche-Z. 2022) For $f \in \mathbb{K}(x)$ the component f_T is Mahler summable if and only if $\mathrm{dres}(f, \tau, k) = 0$ (the zero vector) for each $\tau \in \mathcal{T}_M$ and $k \in \mathbb{N}$.

Proof idea. Similar to the Laurent polynomial case, one adds to f_τ a sequence of "small" summable elements until one obtains a "remainder"

$$\bar{f}_\tau = \sum_{k \geq 1} \sum_{\alpha \in \tau} \frac{\mathrm{dres}(f, \tau, k)_\alpha}{(x - \alpha)^k}$$

such that f_τ is Mahler summable iff $\bar{f}_\tau = 0$.

- The definition (and proofs) for Mahler discrete residues at $\tau(\zeta)$ for $\zeta \in \mathbb{K}_t^\times$ a root of unity is similar in spirit, but more technical, due to the perverse (pre-)periodic behavior of roots of unity under the p-power map.

Main Result

Theorem (Arreche-Z. 2022) Given $f \in \mathbb{K}(x)$. Then f is Mahler summable if and only if $\mathrm{dres}(f, \infty) = 0$ and $\mathrm{dres}(f, \tau, k) = 0$ for all $k \in \mathbb{N}$ and $\tau \in \mathcal{T}_M$.

Thanks!