Mahler Discrete Residues and Summability for Rational Functions

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Joint work with Carlos E. Arreche



Linear Mahler equations

Let \mathbb{K} be an algebraically closed field of char 0, x be an indeterminate, and $p \in \mathbb{Z}_{\geq 2}$.

Consider

$$\ell_r(x)y(x^{p^r}) + \ell_{r-1}(x)y(x^{p^{r-1}}) + \dots + \ell_0(x)y(x) = f(x), \quad (1)$$

where $\ell_i, f \in \mathbb{K}[x]$ are given, y(x) is unknown. A solution of (1) is called a Mahler function.

(Mahler 1929): study Mahler equations to prove the transcendence of values of some functions.

Fact: the generating series of any *p*-automatic sequence (such as the Baum–Sweet and the Rudin–Shapiro sequences) is a Mahler function.

Differential Galois theory

Using differential Galois theory, we can determine the differential-algebraic relations between solutions of Mahler equations.

Example (Roques 2018): A Galoisian proof that the generating series of the Baum-Sweet and Rudin-Shapiro sequences are algebraic independent over $\overline{\mathbb{Q}}(x)$.

Goal: Design effective algorithms for computing differential Galois groups of a given linear Mahler equations.

Endow $\mathbb{K}(x)$ with one of the $\sigma\delta$ -field structures:

(S)
$$\sigma: f(x) \mapsto f(x+1)$$
 and $\delta = \frac{d}{dx}$;

(Q) $\sigma: f(x) \mapsto f(qx)$ with $q \in \mathbb{K}^{\times}$ not root of unity and $\delta = x \frac{d}{dx}$.

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$$\sigma(z_i) = a_i z_i$$
 for some $a_1, \ldots, a_n \in \mathbb{K}(x)^{ imes}$.

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Proposition (Hardouin-Singer 2008) z_1, \ldots, z_n are δ -dependent over $\mathbb{K}(x)$ iff $\exists \mathcal{L}_1, \ldots, \mathcal{L}_n \in \mathbb{K}[\delta]$, linear δ -operators with coefficients in \mathbb{K} , not all 0, and $g \in \mathbb{K}(x)$:

$$\mathcal{L}_1\left(\frac{\delta(a_1)}{a_1}\right) + \cdots + \mathcal{L}_n\left(\frac{\delta(a_n)}{a_n}\right) = \sigma(g) - g.$$

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(Arreche 2017, Arreche-Z. 2022): Using (q-)discrete residues, there exist constants $m_1, \ldots, m_n \in \mathbb{K}$, not all 0, such that

$$m_1 \frac{\delta(a_1)}{a_1} + \cdots + m_n \frac{\delta(a_n)}{a_n} = \sigma(g) - g + c$$

for some $g \in \mathbb{K}(x)$ and $c \in \mathbb{K}$ (with c = 0 in case (S)).

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Motivation

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Idea: Develop the notion of Mahler discrete residues and derive an effective version of Hardouin-Singer's Proposition in the Mahler case.

Continuous residues

Let \mathbb{K} be an algebraically closed field of char 0, and let $f(x) \in \mathbb{K}(x)$. Make the partial fraction decomposition

$$f(x) = r(x) + \sum_{\alpha \in \mathbb{K}} \sum_{k \ge 1} \frac{c_{\alpha}(k)}{(x - \alpha)^k},$$

where $r(x) \in \mathbb{K}[x]$ and $c_{\alpha}(k) \in \mathbb{K}$ (almost all 0).

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Then f(x) is rationally integrable, *i.e.*, there exists $g(x) \in \mathbb{K}(x)$ such that g'(x) = f(x), if and only if the (continuous first-order) residues

$$\operatorname{res}(f, \alpha, 1) := c_{\alpha}(1) = 0$$
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Chen and Singer (2012) created a notion of *discrete residues* that plays an analogous role (where *integrability* \mapsto *summability*) for the *shift* ($x \mapsto x + 1$) and *q*-*dilation* ($x \mapsto qx$) difference operators.

Discrete residues: shift case

Rewrite the partial fraction decomposition of $f(x) \in \mathbb{K}(x)$:

$$f(x) = r(x) + \sum_{k \ge 1} \sum_{[\alpha] \in \mathbb{K}/\mathbb{Z}} \sum_{n \in \mathbb{Z}} \frac{c_{\alpha}(k, n)}{(x - \alpha + n)^k}$$

where $r(x) \in \mathbb{K}[x]$, $\alpha \in \mathbb{K}$ is a coset representative for $[\alpha] := \alpha + \mathbb{Z} \in \mathbb{K}/\mathbb{Z}$, and $c_{\alpha}(k, n) \in \mathbb{K}$.

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Proposition (Chen-Singer 2012) f(x) is rationally *q*-summable, *i.e.*, there exists $g(x) \in \mathbb{K}(x)$ such that f(x) = g(qx) - g(x), if and only if q-dres $(f, \infty) = 0$ and q-dres $(f, [\alpha]_q, k) = 0$ for each $[\alpha]_q \in \mathbb{K}^{\times}/q^{\mathbb{Z}}$ and $k \in \mathbb{N}$.

Why use residues?

An advantage of using residues is to answer whether (yes/no) $f(x) \in \mathbb{K}(x)$ is

- rationally integrable: f(x) = g'(x); or
- rationally summable: f(x) = g(x+1) g(x); or
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In the differential case, there is a better way: if $f = \frac{a}{b}$ with $a, b \in \mathbb{K}[x]$, gcd(a, b) = 1, deg(a) < deg(b), and b squarefree, then the roots of the Rothstein-Trager resultant

$$RT(f) := \operatorname{Res}_{x}(a - z \cdot b', b) \in \mathbb{K}[z]$$

are precisely the first-order continuous residues of f(x), which implies f(x) is rationally integrable iff RT(f) is a monomial in z.

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Mahler Summability Problem: given $f(x) \in \mathbb{K}(x)$, decide effectively whether f(x) is Mahler summable.

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Our Goal: Construct a (K-linear) complete obstruction to the Mahler summability of $f(x) \in K(x)$.

More precisely, for the \mathbb{K} -linear map $\Delta : g(x) \mapsto g(x^p) - g(x)$, we wish to construct explicitly a \mathbb{K} -linear map ∇ on $\mathbb{K}(x)$ such that $\operatorname{im}(\Delta) = \operatorname{ker}(\nabla)$, bypassing computation of certificates.

We call ∇ the Mahler reduction operator. Given $f \in \mathbb{K}(x)$, set $\overline{f} = \nabla(f)$. Then f is Mahler summable if and only if $\overline{f} = 0$. The numerators in the partial fraction decomposition of \overline{f} are Mahler discrete residues of f.

Mahler trajectories and Mahler trees

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We denote by \mathbb{Z}/\mathcal{P} the set of maximal trajectories for the action of \mathcal{P} on \mathbb{Z} by multiplication:

 $\mathbb{Z}/\mathcal{P} = \{\{0\}\} \cup \{\{ip^n \mid n \in \mathbb{Z}_{\geq 0}\} \mid i \in \mathbb{Z} \text{ such that } p \nmid i\}.$

The elements $\theta \in \mathbb{Z}/\mathcal{P}$ are pairwise disjoint subsets of \mathbb{Z} whose union is all of \mathbb{Z} .

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We denote by \mathcal{T}_M the set of equivalence classes for the equivalence relation on \mathbb{K}^{\times} defined by $\alpha \sim \gamma$ if and only if $\alpha^{p^s} = \gamma^{p^r}$ for some $r, s \in \mathbb{Z}_{\geq 0}$.

The elements $\tau \in \mathcal{T}_M$, called Mahler trees, are pairwise disjoint subsets of \mathbb{K}^{\times} whose union is all of \mathbb{K}^{\times} . We write $\tau(\alpha)$ for the unique Mahler tree containing $\alpha \in \mathbb{K}^{\times}$.

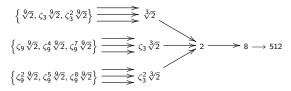
Examples of Mahler trees

We define a digraph on the vertex set τ for each Mahler tree $\tau \in \mathcal{T}_M$ with one directed edge $\alpha \to \gamma$ whenever $\alpha^p = \gamma$.

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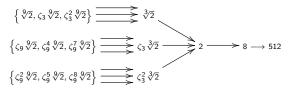
With p = 3, the vertices of $\tau(2)$ near $2 \in \mathbb{K}^{\times}$ are



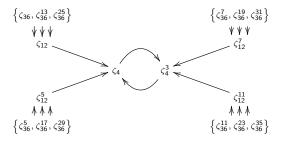
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For $f(x) \in \mathbb{K}(x)$, we can decompose it uniquely as $f(x) = f_L(x) + f_T(x)$:

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Moreover, the decompositions $f_L = \sum_{\theta \in \mathbb{Z}/\mathcal{P}} f_{\theta}$ and $f_T = \sum_{\tau \in \mathcal{T}_M} f_{\tau}$: $f_{\theta} := \sum_{j \in \theta} r_j x^j$ and $f_{\tau} := \sum_{k \ge 1} \sum_{\alpha \in \tau} \frac{c_{\alpha}(k)}{(x - \alpha)^k}$ are also σ -stable. Can decide summability of f by deciding for each

$$f_{ heta} \; (heta \in \mathbb{Z}/\mathcal{P})$$
 and each $f_{ au} \; (au \in \mathcal{T}_{M})$ individually.

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Mahler residues at infinity

Definition (Arreche-Z. 2022) Let $f(x) \in \mathbb{K}(x)$ and write $f_L(x) = \sum_{j \in \mathbb{Z}} r_j x^j \in \mathbb{K}[x, x^{-1}]$. The Mahler residue of f(x) at infinity is the vector

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Proposition (Arreche-Z. 2022) For $f(x) \in \mathbb{K}(x)$ the component $f_L(x) \in \mathbb{K}[x, x^{-1}]$ is Mahler summable if and only if $\operatorname{dres}(f, \infty) = 0$ (the zero vector).

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Definition (Arreche-Z. 2022) Let $f(x) \in \mathbb{K}(x)$ and write $f_L(x) = \sum_{j \in \mathbb{Z}} r_j x^j \in \mathbb{K}[x, x^{-1}]$. The Mahler residue of f(x) at infinity is the vector

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ight)_{ heta\in\mathbb{Z}/\mathcal{P}}\in egin{array}{c} \bigoplus_{ heta\in\mathbb{Z}/\mathcal{P}}\mathbb{K}. \end{array}$$

Proposition (Arreche-Z. 2022) For $f(x) \in \mathbb{K}(x)$ the component $f_L(x) \in \mathbb{K}[x, x^{-1}]$ is Mahler summable if and only if dres $(f, \infty) = 0$ (the zero vector). Proof sketch: For $\theta = \{ip^n\}$ with $p \nmid i$, let $\overline{f}_{\theta}^{(n)} = f_{\theta} + \Delta(g_{\theta}^{(n)})$ with $g_{\theta}^{(0)} := 0$ and $g_{\theta}^{(n+1)} := g_{\theta}^{(n)} + (\sum_{\ell=0}^{n} r_{ip^{\ell}}) \times^{ip^{n}}$. Then, for *h* largest s.t. $r_{ip^{h}} \neq 0$, $\overline{f}^{(h)} = \text{dres}(f, \infty)_{\theta} \cdot x^{ip^{h}}$. A dispersion argument shows $\overline{f}_{\theta}^{(h)} = 0$ iff f_{θ} is Mahler summable.

For $\alpha \in \mathbb{K}^{\times}$, ζ_p a primitive *p*-th root of unity, let $V_k^m(\zeta_p^i \alpha) \in \mathbb{K}$:

$$\sigma\left(\frac{1}{(x-\alpha^p)^m}\right) = \frac{1}{(x^p-\alpha^p)^m} = \sum_{k=1}^m \sum_{i=0}^{p-1} \frac{V_k^m(\zeta_p^i \alpha)}{(x-\zeta_p^i \alpha)^k}.$$

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The definition of Mahler residues is more complicated because applying $\sigma : f(x) \mapsto f(x^p)$ at poles of order m "leaks" into the poles of order $k \leq m$.

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Lemma (Arreche-Z. 2022)
$$V_k^m(\zeta_p^i\alpha) = \mathbb{V}_k^m \cdot \frac{(\zeta_p^i\alpha)^k}{\alpha^{pm}},$$

where $\mathbb{V}_k^m \in \mathbb{Q}$ are obtained from the Taylor coefficients at x = 1: $(x^{p-1} + \dots + x + 1)^{-m} = \sum_{k=1}^m \mathbb{V}_k^m \cdot (x-1)^{m-k} + O((x-1)^m).$

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The "universal coefficients" \mathbb{V}_k^m can be computed directly (as a sum over partitions) using the Faà di Bruno's formula.

Small example of Mahler coefficients

Let
$$p = 3$$
, $m = 2$, and $\alpha^3 = 1$. Then

$$\sigma\left(\frac{1}{(x-1)^2}\right) = \frac{1}{(x^3-1)^2} = \sum_{k=1}^2 \sum_{i=0}^2 \frac{V_k^2(\zeta_3^i)}{(x-\zeta_3^i)^k},$$

By the previous Lemma, $V_k^2(\zeta_3^i) = \mathbb{V}_k^2 \cdot (\zeta_3^i)^{k-6} = \mathbb{V}_k^2 \cdot \zeta_3^{ki}$ for k = 1, 2. We find that

$$\mathbb{V}_2^2 = (x^2 + x + 1)^{-2} \Big|_{x=1} = \frac{1}{9}; \text{ and } \mathbb{V}_1^2 = ((x^2 + x + 1)^{-2})' \Big|_{x=1} = -\frac{2}{9}.$$

Using a computer algebra system (or by hand!), one can very that the partial fraction decomposition of $9 \cdot (x^3 - 1)^{-2}$ is indeed

$$\frac{1}{(x-1)^2} + \frac{\zeta_3^2}{(x-\zeta_3)^2} + \frac{\zeta_3}{(x-\zeta_3^2)^2} + \frac{-2}{x-1} + \frac{-2\zeta_3}{x-\zeta_3} + \frac{-2\zeta_3^2}{x-\zeta_3^2}.$$

Suppose $\gamma \in \mathbb{K}^{\times}$ is not a root of unity, $f \in \mathbb{K}(x)$, and $h \in \mathbb{Z}_{\geq 0}$ s.t.:

 $\operatorname{sing}(f) \cap \tau(\gamma) \subseteq \{\zeta_{p^n}^i \gamma^{p^{h-n}} \mid 0 \le n \le h, \ i \in \mathbb{Z}/p^n\mathbb{Z}\}.$

Mahler residues at Mahler trees (2 of 3): definition Suppose $\gamma \in \mathbb{K}^{\times}$ is not a root of unity, $f \in \mathbb{K}(x)$, and $h \in \mathbb{Z}_{\geq 0}$ s.t.:

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Then we can write, for $au= au(\gamma)$,

$$f_{\tau} = \sum_{k=1}^{m} \sum_{n=0}^{h} \sum_{i \in \mathbb{Z}/p^n \mathbb{Z}} \frac{c_{\gamma}(k,n,i)}{(x-\zeta_{p^n}^i \gamma^{p^{h-n}})^k}.$$

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Set recursively: $\tilde{c}_{k,0,0} := c_{\gamma}(k,0,0)$, and for $1 \le n \le h$; $i \in \mathbb{Z}/p^n\mathbb{Z}$:

$$\tilde{c}_{k,n,i} := c_{\gamma}(k,n,i) + \sum_{j=k}^{m} \tilde{c}_{j,n-1,\pi_{n-1}(i)} V_{k}^{j}(\zeta_{p^{n}}^{i} \gamma^{p^{h-n}}),$$

where $\pi_{n-1}: \mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}/p^{n-1}\mathbb{Z}$ is the canonical projection.

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where $\pi_{n-1} : \mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}/p^{n-1}\mathbb{Z}$ is the canonical projection. Definition (Arreche-Z. 2022) The Mahler discrete residue at τ of order k is the vector dres $(f, \tau, k) \in \bigoplus_{\alpha \in \tau} \mathbb{K}$ with α -component := 0 except possibly at $\alpha = \zeta_{p^h}^i \gamma$ for $i \in \mathbb{Z}/p^h\mathbb{Z}$, given by $\tilde{c}_{k,h,i}$.

Mahler residues at Mahler trees (3 of 3): proof

Proposition (Arreche-Z. 2022) For $f \in \mathbb{K}(x)$ the component f_T is Mahler summable if and only if $\operatorname{dres}(f, \tau, k) = 0$ (the zero vector) for each $\tau \in \mathcal{T}_M$ and $k \in \mathbb{N}$.

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Proof idea. Similar to the Laurent polynomial case, one adds to f_{τ} a sequence of "small" summable elements until one obtains a "remainder"

$$ar{f}_{ au} = \sum_{k \geq 1} \sum_{lpha \in au} rac{\mathrm{dres}(f, au,k)_{lpha}}{(x-lpha)^k}$$

such that f_{τ} is Mahler summable iff $\bar{f}_{\tau} = 0$.

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The definition (and proofs) for Mahler discrete residues at τ(ζ) for ζ ∈ ℝ[×]_t a root of unity is similar in spirit, but more technical, due to the perverse (pre-)periodic behavior of roots of unity under the *p*-power map.

Main Result

Theorem (Arreche-Z. 2022) Given $f \in \mathbb{K}(x)$. Then f is Mahler summable if and only if $\operatorname{dres}(f, \infty) = 0$ and $\operatorname{dres}(f, \tau, k) = 0$ for all $k \in \mathbb{N}$ and $\tau \in \mathcal{T}_M$.

Thanks!