# Mahler Discrete Residues and Summability for Rational Functions

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Joint work with Carlos E. Arreche



### **Linear Mahler equations**

Let  $\mathbb{K}$  be an algebraically closed field of char 0, x be an indeterminate, and  $p \in \mathbb{Z}_{\geq 2}$ .

Consider

$$\ell_r(x)y(x^{p^r}) + \ell_{r-1}(x)y(x^{p^{r-1}}) + \dots + \ell_0(x)y(x) = f(x), \quad (1)$$

where  $\ell_i, f \in \mathbb{K}[x]$  are given, y(x) is unknown. A solution of (1) is called a Mahler function.

(Mahler 1929): study Mahler equations to prove the transcendence of values of some functions.

Fact: the generating series of any *p*-automatic sequence (such as the Baum–Sweet and the Rudin–Shapiro sequences) is a Mahler function.

# **Differential Galois theory**

Using differential Galois theory, we can determine the differential-algebraic relations between solutions of Mahler equations.

Example (Roques 2018): A Galoisian proof that the generating series of the Baum-Sweet and Rudin-Shapiro sequences are algebraic independent over  $\overline{\mathbb{Q}}(x)$ .

Goal: Design effective algorithms for computing differential Galois groups of a given linear Mahler equations.

Endow  $\mathbb{K}(x)$  with one of the  $\sigma\delta$ -field structures:

(S) 
$$\sigma: f(x) \mapsto f(x+1)$$
 and  $\delta = \frac{d}{dx}$ ;

(Q)  $\sigma: f(x) \mapsto f(qx)$  with  $q \in \mathbb{K}^{\times}$  not root of unity and  $\delta = x \frac{d}{dx}$ .

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$$\sigma(z_i) = a_i z_i$$
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Proposition (Hardouin-Singer 2008)  $z_1, \ldots, z_n$  are  $\delta$ -dependent over  $\mathbb{K}(x)$  iff  $\exists \mathcal{L}_1, \ldots, \mathcal{L}_n \in \mathbb{K}[\delta]$ , linear  $\delta$ -operators with coefficients in  $\mathbb{K}$ , not all 0, and  $g \in \mathbb{K}(x)$ :

$$\mathcal{L}_1\left(\frac{\delta(a_1)}{a_1}\right) + \cdots + \mathcal{L}_n\left(\frac{\delta(a_n)}{a_n}\right) = \sigma(g) - g.$$

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(Arreche 2017, Arreche-Z. 2022): Using (q-)discrete residues, there exist constants  $m_1, \ldots, m_n \in \mathbb{K}$ , not all 0, such that

$$m_1 \frac{\delta(a_1)}{a_1} + \cdots + m_n \frac{\delta(a_n)}{a_n} = \sigma(g) - g + c$$

for some  $g \in \mathbb{K}(x)$  and  $c \in \mathbb{K}$  (with c = 0 in case (S)).

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### **Motivation**

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Idea: Develop the notion of Mahler discrete residues and derive an effective version of Hardouin-Singer's Proposition in the Mahler case.

### **Continuous residues**

Let  $\mathbb{K}$  be an algebraically closed field of char 0, and let  $f(x) \in \mathbb{K}(x)$ . Make the partial fraction decomposition

$$f(x) = r(x) + \sum_{\alpha \in \mathbb{K}} \sum_{k \ge 1} \frac{c_{\alpha}(k)}{(x - \alpha)^k},$$

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Then f(x) is rationally integrable, *i.e.*, there exists  $g(x) \in \mathbb{K}(x)$  such that g'(x) = f(x), if and only if the (continuous first-order) residues

$$\operatorname{res}(f, \alpha, 1) := c_{\alpha}(1) = 0$$
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Chen and Singer (2012) created a notion of *discrete residues* that plays an analogous role (where *integrability*  $\mapsto$  *summability*) for the *shift* ( $x \mapsto x + 1$ ) and *q*-*dilation* ( $x \mapsto qx$ ) difference operators.

#### Discrete residues: shift case

Rewrite the partial fraction decomposition of  $f(x) \in \mathbb{K}(x)$ :

$$f(x) = r(x) + \sum_{k \ge 1} \sum_{[\alpha] \in \mathbb{K}/\mathbb{Z}} \sum_{n \in \mathbb{Z}} \frac{c_{\alpha}(k, n)}{(x - \alpha + n)^k}$$

where  $r(x) \in \mathbb{K}[x]$ ,  $\alpha \in \mathbb{K}$  is a coset representative for  $[\alpha] := \alpha + \mathbb{Z} \in \mathbb{K}/\mathbb{Z}$ , and  $c_{\alpha}(k, n) \in \mathbb{K}$ .

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## Why use residues?

An advantage of using residues is to answer whether (yes/no)  $f(x) \in \mathbb{K}(x)$  is

- rationally integrable: f(x) = g'(x); or
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In the differential case, there is a better way: if  $f = \frac{a}{b}$  with  $a, b \in \mathbb{K}[x]$ , gcd(a, b) = 1, deg(a) < deg(b), and b squarefree, then the roots of the Rothstein-Trager resultant

$$RT(f) := \operatorname{Res}_{x}(a - z \cdot b', b) \in \mathbb{K}[z]$$

are precisely the first-order continuous residues of f(x), which implies f(x) is rationally integrable iff RT(f) is a monomial in z.

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Mahler Summability Problem: given  $f(x) \in \mathbb{K}(x)$ , decide effectively whether f(x) is Mahler summable.

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Our Goal: Construct a (K-linear) complete obstruction to the Mahler summability of  $f(x) \in K(x)$ .

More precisely, for the  $\mathbb{K}$ -linear map  $\Delta : g(x) \mapsto g(x^p) - g(x)$ , we wish to construct explicitly a  $\mathbb{K}$ -linear map  $\nabla$  on  $\mathbb{K}(x)$  such that  $\operatorname{im}(\Delta) = \operatorname{ker}(\nabla)$ , bypassing computation of certificates.

We call  $\nabla$  the Mahler reduction operator. Given  $f \in \mathbb{K}(x)$ , set  $\overline{f} = \nabla(f)$ . Then f is Mahler summable if and only if  $\overline{f} = 0$ . The numerators in the partial fraction decomposition of  $\overline{f}$  are Mahler discrete residues of f.

## Mahler trajectories and Mahler trees

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We denote by  $\mathbb{Z}/\mathcal{P}$  the set of maximal trajectories for the action of  $\mathcal{P}$  on  $\mathbb{Z}$  by multiplication:

 $\mathbb{Z}/\mathcal{P} = \{\{0\}\} \cup \{\{ip^n \mid n \in \mathbb{Z}_{\geq 0}\} \mid i \in \mathbb{Z} \text{ such that } p \nmid i\}.$ 

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We denote by  $\mathcal{T}_M$  the set of equivalence classes for the equivalence relation on  $\mathbb{K}^{\times}$  defined by  $\alpha \sim \gamma$  if and only if  $\alpha^{p^s} = \gamma^{p^r}$  for some  $r, s \in \mathbb{Z}_{\geq 0}$ .

The elements  $\tau \in \mathcal{T}_M$ , called Mahler trees, are pairwise disjoint subsets of  $\mathbb{K}^{\times}$  whose union is all of  $\mathbb{K}^{\times}$ . We write  $\tau(\alpha)$  for the unique Mahler tree containing  $\alpha \in \mathbb{K}^{\times}$ .

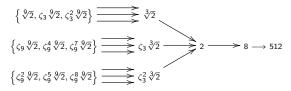
### **Examples of Mahler trees**

We define a digraph on the vertex set  $\tau$  for each Mahler tree  $\tau \in \mathcal{T}_M$  with one directed edge  $\alpha \to \gamma$  whenever  $\alpha^p = \gamma$ .

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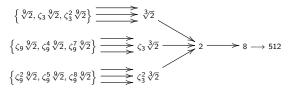
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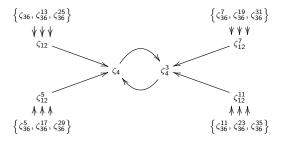
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For  $f(x) \in \mathbb{K}(x)$ , we can decompose it uniquely as  $f(x) = f_L(x) + f_T(x)$ :

$$f_L(x) := \sum_{j \in \mathbb{Z}} r_j x^j$$
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This yields a  $\sigma$ -stable  $\mathbb{K}$ -vector space decomposition of  $\mathbb{K}(x)$  as

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Then  $f \in \mathbb{K}(x)$  is Mahler summable if and only if  $f_L$  and  $f_T$  are both Mahler summable. We address each component separately.

Moreover, the decompositions  $f_L = \sum_{\theta \in \mathbb{Z}/\mathcal{P}} f_{\theta}$  and  $f_T = \sum_{\tau \in \mathcal{T}_M} f_{\tau}$ :  $f_{\theta} := \sum_{j \in \theta} r_j x^j$  and  $f_{\tau} := \sum_{k \ge 1} \sum_{\alpha \in \tau} \frac{c_{\alpha}(k)}{(x - \alpha)^k}$ are also  $\sigma$ -stable. Can decide summability of f by deciding for each

$$f_{ heta} \; ( heta \in \mathbb{Z}/\mathcal{P})$$
 and each  $f_{ au} \; ( au \in \mathcal{T}_{M})$  individually.

Yi Zhang, XJTLU

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#### Mahler residues at infinity

Definition (Arreche-Z. 2022) Let  $f(x) \in \mathbb{K}(x)$  and write  $f_L(x) = \sum_{j \in \mathbb{Z}} r_j x^j \in \mathbb{K}[x, x^{-1}]$ . The Mahler residue of f(x) at infinity is the vector

$$\mathrm{dres}(f,\infty):=\left(\sum_{j\in heta}r_j
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Proposition (Arreche-Z. 2022) For  $f(x) \in \mathbb{K}(x)$  the component  $f_L(x) \in \mathbb{K}[x, x^{-1}]$  is Mahler summable if and only if dres $(f, \infty) = 0$  (the zero vector). Proof sketch: For  $\theta = \{ip^n\}$  with  $p \nmid i$ , let  $\overline{f}_{\theta}^{(n)} = f_{\theta} + \Delta(g_{\theta}^{(n)})$  with  $g_{\theta}^{(0)} := 0$  and  $g_{\theta}^{(n+1)} := g_{\theta}^{(n)} + (\sum_{\ell=0}^{n} r_{ip^{\ell}}) \times^{ip^{n}}$ . Then, for *h* largest s.t.  $r_{ip^{h}} \neq 0$ ,  $\overline{f}^{(h)} = \text{dres}(f, \infty)_{\theta} \cdot x^{ip^{h}}$ . A dispersion argument shows  $\overline{f}_{\theta}^{(h)} = 0$  iff  $f_{\theta}$  is Mahler summable.

For  $\alpha \in \mathbb{K}^{\times}$ ,  $\zeta_p$  a primitive *p*-th root of unity, let  $V_k^m(\zeta_p^i \alpha) \in \mathbb{K}$ :

$$\sigma\left(\frac{1}{(x-\alpha^p)^m}\right) = \frac{1}{(x^p-\alpha^p)^m} = \sum_{k=1}^m \sum_{i=0}^{p-1} \frac{V_k^m(\zeta_p^i \alpha)}{(x-\zeta_p^i \alpha)^k}.$$

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The definition of Mahler residues is more complicated because applying  $\sigma : f(x) \mapsto f(x^p)$  at poles of order m "leaks" into the poles of order  $k \leq m$ .

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Lemma (Arreche-Z. 2022)  
$$V_k^m(\zeta_p^i\alpha) = \mathbb{V}_k^m \cdot \frac{(\zeta_p^i\alpha)^k}{\alpha^{pm}},$$

where  $\mathbb{V}_k^m \in \mathbb{Q}$  are obtained from the Taylor coefficients at x = 1:  $(x^{p-1} + \dots + x + 1)^{-m} = \sum_{k=1}^m \mathbb{V}_k^m \cdot (x-1)^{m-k} + O((x-1)^m).$ 

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The "universal coefficients"  $\mathbb{V}_k^m$  can be computed directly (as a sum over partitions) using the Faà di Bruno's formula.

#### Small example of Mahler coefficients

Let 
$$p = 3$$
,  $m = 2$ , and  $\alpha^3 = 1$ . Then

$$\sigma\left(\frac{1}{(x-1)^2}\right) = \frac{1}{(x^3-1)^2} = \sum_{k=1}^2 \sum_{i=0}^2 \frac{V_k^2(\zeta_3^i)}{(x-\zeta_3^i)^k},$$

By the previous Lemma,  $V_k^2(\zeta_3^i) = \mathbb{V}_k^2 \cdot (\zeta_3^i)^{k-6} = \mathbb{V}_k^2 \cdot \zeta_3^{ki}$  for k = 1, 2. We find that

$$\mathbb{V}_2^2 = (x^2 + x + 1)^{-2} \Big|_{x=1} = \frac{1}{9}; \text{ and } \mathbb{V}_1^2 = ((x^2 + x + 1)^{-2})' \Big|_{x=1} = -\frac{2}{9}.$$

Using a computer algebra system (or by hand!), one can very that the partial fraction decomposition of  $9 \cdot (x^3 - 1)^{-2}$  is indeed

$$\frac{1}{(x-1)^2} + \frac{\zeta_3^2}{(x-\zeta_3)^2} + \frac{\zeta_3}{(x-\zeta_3^2)^2} + \frac{-2}{x-1} + \frac{-2\zeta_3}{x-\zeta_3} + \frac{-2\zeta_3^2}{x-\zeta_3^2}.$$

Suppose  $\gamma \in \mathbb{K}^{\times}$  is not a root of unity,  $f \in \mathbb{K}(x)$ , and  $h \in \mathbb{Z}_{\geq 0}$  s.t.:

 $\operatorname{sing}(f) \cap \tau(\gamma) \subseteq \{\zeta_{p^n}^i \gamma^{p^{h-n}} \mid 0 \le n \le h, \ i \in \mathbb{Z}/p^n\mathbb{Z}\}.$ 

#### Mahler residues at Mahler trees (2 of 3): definition Suppose $\gamma \in \mathbb{K}^{\times}$ is not a root of unity, $f \in \mathbb{K}(x)$ , and $h \in \mathbb{Z}_{\geq 0}$ s.t.:

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Then we can write, for  $au= au(\gamma)$ ,

$$f_{\tau} = \sum_{k=1}^{m} \sum_{n=0}^{h} \sum_{i \in \mathbb{Z}/p^n \mathbb{Z}} \frac{c_{\gamma}(k,n,i)}{(x-\zeta_{p^n}^i \gamma^{p^{h-n}})^k}.$$

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Set recursively:  $\tilde{c}_{k,0,0} := c_{\gamma}(k,0,0)$ , and for  $1 \le n \le h$ ;  $i \in \mathbb{Z}/p^n\mathbb{Z}$ :

$$\tilde{c}_{k,n,i} := c_{\gamma}(k,n,i) + \sum_{j=k}^{m} \tilde{c}_{j,n-1,\pi_{n-1}(i)} V_{k}^{j}(\zeta_{p^{n}}^{i} \gamma^{p^{h-n}}),$$

where  $\pi_{n-1}: \mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}/p^{n-1}\mathbb{Z}$  is the canonical projection.

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where  $\pi_{n-1} : \mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}/p^{n-1}\mathbb{Z}$  is the canonical projection. Definition (Arreche-Z. 2022) The Mahler discrete residue at  $\tau$  of order k is the vector dres $(f, \tau, k) \in \bigoplus_{\alpha \in \tau} \mathbb{K}$  with  $\alpha$ -component := 0 except possibly at  $\alpha = \zeta_{p^h}^i \gamma$  for  $i \in \mathbb{Z}/p^h\mathbb{Z}$ , given by  $\tilde{c}_{k,h,i}$ .

#### Mahler residues at Mahler trees (3 of 3): proof

Proposition (Arreche-Z. 2022) For  $f \in \mathbb{K}(x)$  the component  $f_T$  is Mahler summable if and only if  $\operatorname{dres}(f, \tau, k) = 0$  (the zero vector) for each  $\tau \in \mathcal{T}_M$  and  $k \in \mathbb{N}$ .

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Proof idea. Similar to the Laurent polynomial case, one adds to  $f_{\tau}$  a sequence of "small" summable elements until one obtains a "remainder"

$$ar{f}_{ au} = \sum_{k \geq 1} \sum_{lpha \in au} rac{\mathrm{dres}(f, au,k)_{lpha}}{(x-lpha)^k}$$

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The definition (and proofs) for Mahler discrete residues at τ(ζ) for ζ ∈ ℝ<sup>×</sup><sub>t</sub> a root of unity is similar in spirit, but more technical, due to the perverse (pre-)periodic behavior of roots of unity under the *p*-power map.

#### **Main Result**

Theorem (Arreche-Z. 2022) Given  $f \in \mathbb{K}(x)$ . Then f is Mahler summable if and only if  $\operatorname{dres}(f, \infty) = 0$  and  $\operatorname{dres}(f, \tau, k) = 0$  for all  $k \in \mathbb{N}$  and  $\tau \in \mathcal{T}_M$ .

### Thanks!